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ANNALS OF MATHEMATICS

(FOUNDED BY ORMOND STONE)

EDITED BY
ORMOND STONE W. E. BYERLY H. S. WHITE
W. F. OSGOOD F. S. WOODS

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ON THE CONVERGENCE OF THE CONTINUED FRACTION OF GAUSS AND OTHER CONTINUED FRACTIONS.*

BY EDWARD B. VAN VLECK.

IN the *Disquisitiones generales circa seriem infinitam* . . . of Gauss the quotient

$$G(\alpha, \beta, \gamma, x) = \frac{F(\alpha, \beta + 1, \gamma + 1, x)}{F(\alpha, \beta, \gamma, x)}$$

was developed into the continued fraction

$$\frac{1}{1} - \frac{c_2 x}{1} - \frac{c_3 x}{1} - \ldots,$$

in which

$$c_{2n} = \frac{(\alpha + n - 1)(\gamma - \beta + n - 1)}{(\gamma + 2n - 2)(\gamma + 2n - 1)}, \quad c_{2n+1} = \frac{(\beta + n)(\gamma - \alpha + n)}{(\gamma + 2n - 1)(\gamma + 2n)}.$$

The convergence of this continued fraction was later a subject of investigation by Thomé† and Riemann.‡ Both writers showed that, if the portion of the real axis between $x = 1$ and $x = +\infty$ is regarded as a cut, the continued fraction will converge in the remainder of the plane of x except at the points in which $F(\alpha, \beta, \gamma, x)$ and its analytic continuation vanish, and that the limit of the continued fraction is the analytic function defined by $G(\alpha, \beta, \gamma, x)$.§ The proofs of Thomé and Riemann are both of a special and somewhat involved character, not based upon considerations which can be applied to prove the convergence of any considerable class of continued fractions. Indeed, it is only within the past decennium that general theorems for the convergence of algebraic[*] continued fractions[†] have been obtained. The object of this paper is to demonstrate the convergence of the continued fraction of Gauss by theorems which are of the latter character. The theorems employed for this purpose are new and cover a wide class of algebraic continued fractions. Further application of these theorems is made to the quotient of two related Bessel's functions, J_{n-1}/J_n, and to the continued fraction which Heine has given as a generalization of that of Gauss.

---

* This paper was read before the American Mathematical Society at its summer meeting in Ithaca, Aug. 20, 1901.

† *Crelle*, vol. 67 (1867), p. 299.

‡ Sullo svolgimento del quoziente di due serie ipergeometriche in frazione continua infinita, *Gesammelte Werke*, 1st ed., p. 400. The completion of this posthumous fragment is due to Schwarz.

§ An elementary proof for the special case in which $0 < x < 1$ is found in Schlömilch, *Algebr. Analysis*, p. 321.

I. THE FUNDAMENTAL THEOREMS.

1. *Statement of the First Two Theorems.* To most of the common algebraic continued fractions one or both of the two following theorems may be applied:

THEOREM 1. *If k denotes the greatest modulus of a point of condensation of the coefficients a_n of the continued fraction*

$$\frac{a_1}{1+}\,\frac{a_2x}{1+}\,\frac{a_3x}{1+}\cdots\cdot, \tag{1}$$

the continued fraction will represent an analytic function within a circle of radius $1/4k$, described about the origin of the x-plane as centre, and the only singularities of the function contained within the circle will be poles.

If, furthermore, any circle of smaller radius is drawn about the origin and from this circle each of the poles is excluded by drawing around it a small but arbitrary contour, then within the region remaining the continued fraction will converge uniformly to the analytic function as its limit.

THEOREM 2. *If from and after some fixed point in the continued fraction*

$$\frac{1}{b_1z+}\,\frac{1}{b_2+}\,\frac{1}{b_3z+}\,\frac{1}{b_4+}\cdots\cdot\cdot \tag{2}$$

b_n is real and positive,[‡] *and $\sum\limits_{n=1}^{\infty} b_n$ is divergent, the continued fraction will converge over the entire plane of z with the exception of*

(1) *the whole or a part of the negative half of the real axis;*

---

[*] Continued fractions whose partial quotients are functions of one or more variables.

[†] A summary of the few theorems obtained hitherto will be found in the *Transactions of the American Mathematical Society*, vol. 2, 1901, p. 215.

[‡] The theorem holds also if from and after some fixed point b_n is negative.

(2) *a series of isolated points* $p_1, p_2, \ldots,$ *which lie without this half of the axis.*[*]

The convergence, moreover, is uniform in any region from which these points and the negative half-axis are excluded after the manner of Theorem 1, *and the limit of the continued fraction is an analytic function whose only singularities exterior to the negative half-axis are the points* $p_1, p_2, \ldots,$ *which are poles.*

2. *Proof of these Theorems.* The proof of Theorem 1 will be found in the *Transactions of the American Mathematical Society*, vol. 2, October, 1901. Theorem 2 may be demonstrated as follows.

By hypothesis, there is some point in (2) subsequent to which b_n is real and positive. Suppose this to be true when $n \geqq 2\mu$, and consider the fraction

$$\frac{1}{b_{2\mu+1}z} + \frac{1}{b_{2\mu+2}} + \frac{1}{b_{2\mu+3}z} + \cdots. \tag{3}$$

If the nth convergent of (2) is denoted by N_n/D_n and the mth convergent of (3) by N'_m/D'_m, we obviously have

$$\frac{N_{2\mu+m}}{D_{2\mu+m}} = \frac{N_{2\mu} + \dfrac{N'_m}{D'_m} N_{2\mu-1}}{D_{2\mu} + \dfrac{N'_m}{D'_m} D_{2\mu-1}}. \tag{4}$$

Hence to ascertain whether the successive convergents of (2) approach a limit, it is sufficient first to determine whether this is the case with the convergents of (3).

Now Stieltjes[†] has proved that when the coefficients $b_{2\mu+n}$ in the continued fraction (3) are positive for all values of n and $\Sigma b_{2\mu+n}$ is divergent, the continued fraction will converge uniformly in any region T, bounded by a closed curve which has no point in common with the negative half of the axis of z, and the limit of the fraction is a function which is holomorphic in T. It follows that when m is indefinitely increased, N'_m/D'_m converges uniformly within any such region, and its limit $\phi(z)$ is holomorphic at every point without the negative half-axis. Since the numerator and denominator of the right hand member of (4) also converge in the same manner, their quotient must converge uniformly to the limit

---

[*] It is proved later (§3) that if b_n fulfills the conditions of the theorem and is also real from the beginning of the continued fraction, the number of the points p_n must be finite. Whether this number is always finite does not appear; but if the number is infinite, the derivative of the point-set obviously can consist only of points which lie upon the negative half of the axis.

[†] *Annales de Toulouse*, vol. 8. Another proof is given by the writer in the *Transactions Amer. Math. Society*, vol. 2, 1901; see in particular p. 231–2.

$$\frac{N_{2\mu} + \phi(z)\, N_{2\mu-1}}{D_{2\mu} + \phi(z)\, D_{2\mu-1}} \tag{5}$$

except in the vicinity of the roots of $D_{2\mu} + \phi(z)\, D_{2\mu-1}$ or along the negative half of the axis. It is impossible for $D_{2\mu} + \phi(z)\, D_{2\mu-1}$ to vanish identically, since in that case the numerator would have to vanish identically too, and hence $N_{2\mu} D_{2\mu-1} - N_{2\mu-1} D_{2\mu} \equiv 0$, which is not true. As the numerator and denominator of (5) are also holomorphic at all points without the half-axis, the expression (5) is an analytic function whose only singularities exterior to the half-axis are poles. The continued fraction (2) therefore converges in the manner stated in the theorem.*

3. *A Sufficient Condition that the Number of the Poles be Finite.* In at least one case (which is stated below in Theorem 3) we can easily prove that the number of poles $p_1, p_2, \ldots$ is finite.

When, namely, the coefficients in the continued fraction (2) are real for all values of n and satisfy the conditions of Theorem 2, an upper limit can be found for the number of poles of the limiting function exterior to the negative half of the axis. These poles, in fact, correspond to the zeros of the analytic function to which D_n converges as n increases indefinitely, and the convergence is obviously uniform in the vicinity of each zero which does not lie upon the negative half-axis. Now it is well known that when a series of functions $f_n(z)$, holomorphic throughout any region T, converges uniformly in this region, the zeros of the limiting function within T are the points of condensation of the zeros of the functions $f_n(z)$. Here we have $f_n(z) = D_{2\mu} + \frac{N_n'}{D_n'} D_{2\mu-1}$. Furthermore, Hurwitz† has shown that if a circle of sufficiently small radius is drawn about each zero of the limiting function within T, each of the circles will contain a number of roots of $f_n(z)$ exactly equal to the order of the zero enclosed, for all values of $n \geqq m$, where m is a fixed integer. Our question concerning the number of poles is thus reduced to a determination of an upper limit for the number of roots of D_n exterior to the negative half of the axis.

---

* For the sake of completeness it may be added that when $\Sigma\, b_n$ is convergent, the even convergents of (2) and the odd convergents approach separate limits, so that it is impossible to speak of the convergence of the continued fraction or of a single limiting function which it represents. This is always the case when Σb_n converges, irrespective of other conditions which may be imposed on b_n, and the limits of the two series of convergents are meromorphic over the entire plane. This result is not new. See, for example, the memoir of Stieltjes previously cited. For another mode of proof see the *Transactions Amer. Math. Soc.*, vol. 2, 1901, p. 231.

† *Math. Annalen*, vol. 33, 1888, p. 249.

Let ν be the smallest value of n subsequent to which b_n is always positive, and consider then the number of changes of sign in the series

$$D_\nu, \quad D_{\nu+2}, \quad D_{\nu+4}, \quad \ldots, \quad D_{\nu+2n} \tag{6}$$

for any real value of z. When z varies continuously, this number will change only when one or more of the D's vanish. From the fundamental relation connecting the denominators of three successive convergents it will be found that

$$D_n = \left(b_n b_{n-1} z + 1 + \frac{b_n}{b_{n-2}}\right) D_{n-2} - \frac{b_n}{b_{n-2}} D_{n-4}, \tag{7}$$

alike when n is odd or even. Accordingly, when any term in (6) vanishes, the preceding and following terms have opposite signs. If two successive terms should vanish, then simultaneously every $D_{\nu \pm 2i}$ vanishes. But $D_0 \equiv 1$, $D_1 \equiv b_{2\mu+1} z$. This happens therefore only when ν is odd and $z = 0$. Hence we see that an alteration in the number of changes of sign in (6) can take place only when z passes through a root of D_ν or $D_{\nu+2n}$, and that there is a loss or a gain of a single sign for a root of either unless $z = 0$.

Suppose now that ν is even. If we throw out a common factor, the terms of highest degree in the successive members of (6) will be equal to

$$1, \quad b_{\nu+1} b_{\nu+2} z, \quad b_{\nu+1} b_{\nu+2} b_{\nu+3} b_{\nu+4} z^2, \quad \ldots\ldots, \quad b_{\nu+1} \cdot\cdot\, b_{\nu+2n} z^n. \tag{8}$$

When, therefore, $z = -\infty$ is substituted in (6), there will be n alterations of sign. On the other hand, the terms of (6) all have the same sign when $z = 0$, since the constant term of D_{2n} is always equal to 1. It follows that D_ν and $D_{\nu+2n}$ together must have at least n roots in the negative half-axis. But their degrees are equal respectively to $\nu/2$ and $n + \nu/2$. We conclude therefore that the number of roots of $D_{\nu+2n}$ without the half-axis can not exceed ν.

Suppose next that ν is odd. As before, there will be n changes of sign in (6) when $z = -\infty$. The terms of lowest degree, however, are equal now to

$$(b_1 + b_3 + b_5 + \cdots + b_\nu)z, \quad (b_1 + b_3 + \cdots + b_{\nu+2})z, \cdots, \quad (b_1 + b_3 + \cdots + b_{\nu+2n})z.$$

Since b_n, by hypothesis, is positive for $n > \nu$, the number of changes of sign here can not exceed unity. Hence the number of changes of sign in (6) for an exceedingly small negative value of z cannot exceed unity, and D_ν and $D_{\nu+2n}$ together have at least $n - 1$ roots between $z = 0$ and $z = -\infty$. As they also each have a root in the origin and as their degrees are equal respectively to $(\nu + 1)/2$ and $n + (\nu + 1)/2$, we conclude again that the number of roots of D_n exterior to the negative half of the axis will not exceed ν. We reach thus the following result:

THEOREM 3. *If the coefficients b_n are real from the outset in the continued fraction of Theorem 2 and if ν is the smallest value of n subsequent to which they are always positive, the number of poles of the function defined by the continued fraction, exterior to the negative half of the real axis of z, will not exceed ν, each multiple pole being counted a number of times equal to its order.*

4. *A Sufficient Condition for the Applicability of Theorem* 1. We proceed now to determine a class of continued fractions of the form (1) to which Theorem 1 is applicable. In many continued fractions of that form, it is possible, for sufficiently large values of n, to expand a_{2n-1}/a_{2n} and a_{2n}/a_{2n+1} into convergent series in ascending powers of $1/n$ with constant coefficients. Suppose then that

$$\frac{a_{2n-1}}{a_{2n}} = 1 + \frac{A_1'}{n} + \frac{A_2'}{n^2} + \cdots\cdot, \tag{9}$$

$$\frac{a_{2n}}{a_{2n+1}} = 1 + \frac{A_1''}{n} + \frac{A_2''}{n^2} + \cdots\cdot \tag{10}$$

From these we derive

$$\frac{a_{2n+1}}{a_{2n-1}} = 1 - \frac{A_1' + A_1''}{n} + \frac{1}{n^2}(\quad) + \cdots\cdot, \tag{11}$$

and a similar power series for a_{2n+2}/a_{2n}, in which the first two terms are the same as in (11). If now the real part of $A_1' + A_1''$ is positive, a theorem of Weierstrass* shows that a_{2n+1} and a_{2n} with increasing n approach the limit 0.

---

* Ueber die Theorie der analytischen Facultäten, *Crelle*, vol. 51, 1856, p. 25; *Werke*, vol. 1, p. 181.

Theorem 1 then gives complete information concerning the continued fraction and shows that it represents a function which is either holomorphic or meromorphic over the entire finite plane.

If the real part of $A_1' + A_1''$ is negative, a_n approaches the limit $k = \infty$, and no information at all concerning the continued fraction can be obtained from Theorem 1.

Lastly, if the real part of $A_1' + A_1''$ is equal to zero, the moduli of a_{2n} and a_{2n+1} approach finite limits, which by (10) are seen to be identical. Their arguments, however, will fail to approach definite values, unless the imaginary part of $A_1' + A_1''$ vanishes, too. When this happens, a_{2n} and a_{2n+1} must both approach limiting values, since $\prod\limits_{n=1}^{\infty}(1 + \frac{1}{n^2})$ is convergent. The identity of the two limits then follows from (10). *If, therefore, equations* (9) *and* (10) *hold for large values of n, the necessary and sufficient condition that a_n shall approach a finite limit different from zero is that $A_1' + A_1''$ shall be equal to zero.*

5. *The Convergence of the Continued Fraction* (1) *when* $A_1' + A_1'' = 0$. *Case* I: a_n *Ultimately Real.* When the condition:

$$A_1' + A_1'' = 0$$

is fulfilled, Theorem 1 determines the convergence of the continued fraction for a portion of the x-plane.* Further information regarding its convergence can be gained by throwing it into the form (2). For this purpose put $x = 1/z$ and then transform so as to remove z from the partial numerators. It will then appear in the alternate denominators, beginning with the first. An easy and well known reduction, which alters neither the value of the continued fraction nor the value of its convergents, then gives the following values for the coefficients of (2):

$$b_1 = \frac{1}{a_1}, \quad b_{2n} = \frac{a_1 a_3 \ . \ . \ . \ a_{2n-1}}{a_2 a_4 \ . \ . \ . \ a_{2n}}, \quad b_{2n+1} = \frac{a_2 a_4 \ . \ . \ . \ a_{2n}}{a_1 a_3 \ . \ . \ . \ a_{2n+1}}. \qquad (12)$$

We will take up first the case in which a_n *is real for all values of* $n \geqq m$, where m denotes a fixed number. Since by hypothesis

$$\operatorname*{limit}_{n=\infty} \frac{a_{n-1}}{a_n} = 1,$$

---

* Theorem 1 is also applicable when only the real part $A_1' + A_1''$ vanishes. This case will not concern us here.

a_n must ultimately have a constant sign. If this sign is $-$, it may be changed to $+$ by putting $x = -x'$. We may therefore assume that a_n is real and positive when n exceeds the even integer 2μ.

Let now the first 2μ partial quotients of (1) be omitted, and consider the continued fraction which remains. For convenience we will denote the coefficients of its numerators by $a'_1, a'_2, \cdots$. The corresponding values of b'_{2n} and b'_{2n+1} are real and positive. Furthermore, when n is sufficiently increased, we have

$$\frac{b'_{2n}}{b'_{2n-2}} = \frac{a'_{2n-1}}{a'_{2n}} = 1 + \frac{A'_1}{n+\mu} + \frac{A'_2}{(n+\mu)^2} + \cdots, \tag{13}$$

$$\frac{b'_{2n+1}}{b'_{2n-1}} = \frac{a'_{2n}}{a'_{2n+1}} = 1 + \frac{A''_1}{n+\mu} + \cdots. \tag{14}$$

Now since $A'_1 + A''_1 = 0$, either A'_1 or A''_1 must be greater than -1; say the former. Then

$$\lim_{n=\infty} n\left(\frac{b'_{2n}}{b'_{2n-2}} - 1\right) > -1,$$

whence it follows by a theorem of Raabe's* that $\sum\limits_{n=1}^{\infty} b'_{2n}$ is divergent. If $A''_1 > -1$, we see similarly that $\Sigma\, b'_{2n+1}$ diverges. In either case, all the conditions of the theorem of Stieltjes quoted in §2 are fulfilled, and the continued fraction, after the omission of the first 2μ partial quotients, will converge over the entire plane of z except along a cut from $z = 0$ to $z = -\infty$. Since $x = 1/z$, there is a similar region of convergence in the plane of x. When the 2μ partial quotients are restored, the reasoning of §2 may be repeated. We conclude therefore that under the conditions indicated the continued fraction (1) represents a function whose only singularities exterior to the negative half-axis of x are poles.†

6. *Continuation; Case* II: a_n *Not Ultimately Real.* The direct consideration of (1) when $A'_1 + A''_1 = 0$, appears to be difficult if a_n is not ultimately real, or at least if A_1' and A_1'' are not real. For Weierstrass* shows that the argument of b_n will fail to approach a limiting value if the imaginary component of A_1' in (13) or of A_1'' in (14) is not equal to zero. On this account it is better to proceed indirectly.

---

* *Zeitschrift für Physik und Mathematik*, vol. 10, Vienna, 1832, p. 63; *cf.* for example, Stolz, *Allgemeine Arithmetik*, vol. 1, p. 263.

† If $A'_1 < -1$ and $A''_1 < -1$, the series $\Sigma b'_n$ is convergent. In this case the limit of the series of odd convergents or of the series of even convergents is meromorphic over the entire plane of x, but the two limits are not identical.

Equation (7) enables us to consider the even and the odd convergents separately. Since a similar relation connects the numerators of the convergents of (2), the even convergents form by themselves the successive convergents of the continued fraction

$$\frac{b_2}{b_1b_2z + 1} - \frac{\frac{b_4}{b_2}}{b_4b_3z + 1 + \frac{b_4}{b_2}} - \frac{\frac{b_6}{b_4}}{b_6b_5z + 1 + \frac{b_6}{b_4}} - \cdots, \tag{15}$$

while the odd convergents have a like connection with

$$\frac{1}{b_1z}\left(1 - \frac{\frac{b_3}{b_1}}{b_3b_2z + 1 + \frac{b_3}{b_1}} - \frac{\frac{b_5}{b_3}}{b_5b_4z + 1 + \frac{b_5}{b_3}} - \cdots\right). \tag{16}$$

If in (16) we set $\frac{b_{2n+1}}{b_{2n-1}} = a_n''$ and then reduce the continued fraction to the normal form in which each partial numerator is equal to 1, it becomes

$$\frac{1}{b_1z}\left(1 - \frac{1}{b_1''\left(b_3b_2z + 1 + \frac{b_3}{b_1}\right)} - \frac{1}{b_2''\left(b_5b_4z + 1 + \frac{b_5}{b_3}\right)} - \cdots\right), \tag{17}$$

in which b_n'' is obtained from the a_n'' by doubly accenting the letters of (12).

The continued fraction (17) is of the form

$$1 - \frac{1}{B_1z + C_1} - \frac{1}{B_2z + C_2} - \cdots\cdots \tag{17a}$$

We shall show that B_n approaches a limit different from 0, which we shall denote by k^{-1}, and that C_n approaches 2 as its limit, when $n = \infty$. The convergence of the continued fraction

$$1 - \frac{1}{k^{-1}z + 2} - \frac{1}{k^{-1}z + 2} - \cdots\cdot \tag{17b}$$

---

* *Werke, l. c.*, p. 181-184.

can be dealt with directly by means of theorems already established, and then the fraction (17a) can be treated in a similar manner.

Let the first $2r - 1$ partial quotients of (1) be omitted and consider in its place the continued fraction which remains. If r has been taken sufficiently large we shall have in the continued fraction corresponding to (17)

$$b_1'' = \frac{b_1}{b_3} = \frac{a_{2r+2}}{a_{2r+1}} = 1 - \frac{A_1''}{r+1} + \frac{(\ )}{(r+1)^2} + \cdots,$$

and the value of b_1'' can accordingly be made to differ from unity by as little as we choose. The same is obviously true of b_2''. Furthermore

$$\frac{b_{2n}''}{b_{2n-2}''} = \frac{a_{2n-1}''}{a_{2n}''} = \frac{b_{4n-1}}{b_{4n-3}} \div \frac{b_{4n+1}}{b_{4n-1}} = \frac{a_{4n+2r-3}}{a_{4n+2r-2}} \div \frac{a_{4n+2r-1}}{a_{4n+2r}},$$

whence we obtain with the aid of (9) and (10)

$$\frac{b_{2n}''}{b_{2n-2}''} = \frac{1 + \dfrac{A_1'}{2n+r-1} + \dfrac{A_2'}{(2n+r-1)^2} + \cdots}{1 + \dfrac{A_1'}{2n+r} + \dfrac{A_2'}{(2n+r)^2} + \cdots}$$

$$= 1 + \frac{B_2}{(2n+r)^2} + \frac{B_3}{(2n+r)^3} + \cdots.$$

It will be noticed that the first power of $1/(2n + r)$ does not appear. We can show in like manner that it is absent in the expansion of b_{2n+1}''/b_{2n-1}''. There must therefore exist a positive number C such that the inequality

$$\left|\frac{b_n''}{b_{n-2}''}\right| < 1 + \frac{C}{(n+r)^2}$$

shall hold for all values of n. Now the product

$$\prod_{n=1}^{\infty}\left(1 + \frac{C}{(n+r)^2}\right)$$

by increasing r sufficiently can be made to differ from unity by as little as we please. Since this is also the case with b_1'' and b_2'', it is possible so to choose r as to make

$$|b_n'' - 1| < \epsilon_1, \qquad (n = 1, 2, \ldots),$$

in which ϵ_1 is a small but arbitrarily assigned positive quantity.

Furthermore, in each partial denominator of (17) we have the relation $b_n b_{n+1} = 1/a_{n+2r}$. If, then, k is the limit of a_n, the following condition may also be fulfilled

$$|b_n b_{n+1} - k^{-1}| < \epsilon_1, \qquad (n = 1, 2, \ldots)$$

Lastly, since

$$\frac{b_{n+2}}{b_n} = \frac{a_{n+2r}}{a_{n+2r+1}},$$

the value of this ratio can be made to differ from unity by a quantity of as small a modulus as is desired. Hence we see, finally, that *if r has been taken sufficiently large, the successive partial denominators of* (17) *for any given finite value of z will all differ from $k^{-1}z + 2$ by a quantity whose modulus is smaller than an arbitrarily prescribed quantity ϵ. It is also possible so to choose r that this shall hold uniformly for a given finite region of the plane of z.*

7. *Completion of the Proof in Case* II. We now have all of the material at hand for the final determination of the region of convergence of the continued fraction (1). We suppose that r fulfills the condition of the close of §6 and place

$$w = k^{-1}z.$$

Also let T' be the corresponding region in the plane of w. This region shall lie wholly above or else wholly below the real axis. If the continued fraction (17) is transformed so as to make each of its numerators equal to $+1$, the signs of the alternate denominators must be changed. Hence at each point of T' the imaginary component of the denominators will be alternately positive and negative. Also, in T' the absolute value of the ratio of the imaginary to the real component of any denominator will never sink below a finite minimum. Lastly, the sum of the absolute values of the denominators is divergent throughout T'. When these three conditions are fulfilled, it follows by a theorem which I have given in the *Transactions of the American Mathematical Society*, vol. 2, p. 229, that the continued fraction converges throughout T'. The reasoning of the succeeding pages also shows that the convergence is uniform throughout T'.

We have thus shown, under the conditions stated, that if a sufficient number of partial quotients at the outset in (1) are omitted, the convergents of odd order will converge at each point of T'. A similar course of reasoning applied to (15) will prove that this is also true of the even convergents. Since the convergence is uniform, the limits are holomorphic within T'. If, now, the omitted partial quotients are restored, the reasoning of §2 makes it evident that the limits of the two series of convergents are analytic functions whose only singularities in the interior of T'' are poles. Since $w = k^{-1}x^{-1}$, it follows from Theorem 1 that the limits of the two series of convergents are identical in the portion of the w-plane which lies without a circle described about the origin as center with a radius equal to 4. If T'' does not extend into this portion of the plane, it is possible to enlarge it so that it shall. Thus the limits of the odd and of the even convergents must coincide throughout the whole region T'' since they are analytic functions and coincide throughout a part of this region. Finally T'' can be enlarged so as to include as much of the positive (or of the negative) half-plane as is desired. We conclude therefore that if a cut is made along the real axis from $w = -4$ to $w = 4$, the continued fraction will represent a function which is meromorphic in the remainder of the plane.

To obtain the corresponding region of the x-plane, we first draw a straight line through the points $x = \pm 1/4k$ and then cut the plane along the two segments of the line which lie between these points and ∞.

8. Our last step will be to show that one of the two cuts just made may be dispensed with. Take for this purpose any region T''' in the w-plane in which the real part of w is positive and which includes a portion of the real axis. We have already proved that by properly choosing r the denominators in the continued fraction (17) can be made to differ from $w + 2$ throughout T'' by a quantity whose modulus is smaller than an arbitrarily prescribed quantity ϵ. If ϵ is taken sufficiently small, the moduli of the denominators will exceed 2 at each point of T''', inasmuch as the real part of w is positive. This will also hold for each denominator, except the first, in the continued fraction which we derive from (15) in the same manner as (17) was derived from (16). The first denominator, $b_2^{-1}(b_1 b_2 k w + 1)$, will obviously exceed 1 by at least some small quantity h, if r is sufficiently large. The convergence of both continued fractions within T''' then follows from a theorem of Pringsheim.* We shall, however, reproduce his reasoning here in such form as to make it immediately apparent that the convergence of both fractions is uniform.

For convenience, let the nth partial quotient in either continued fraction be denoted by $1/r_n e^{i\theta_n}$. According to our hypothesis we have for each point of T'

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* *Sitzungsberichte der Münchener Akademie*, vol. 28, 1898, p. 313–316.

$$r_1 > 1 + h,$$

$$r_n \geqq 2 \qquad (n > 1),$$

and hence

$$|D_2| - |D_1| \geqq h.$$

From the relation

$$D_n = r_n e^{i\theta_n} D_{n-1} + D_{n-2}$$

it follows that

$$|D_n| - |D_{n-1}| \geqq |D_{n-1}| - |D_{n-2}| \geqq \cdots \geqq h.$$

Hence $|D_n| \geqq 1 + nh$. Furthermore

$$\frac{N_{n+1}}{D_{n+1}} - \frac{N_n}{D_n} = -\frac{1}{D_n D_{n+1}}$$

and consequently

$$\left|\frac{N_{n+i}}{D_{n+i}} - \frac{N_n}{D_n}\right| \leqq \frac{1}{|D_n D_{n+1}|} + \frac{1}{|D_{n+1} D_{n+2}|} + \cdots + \frac{1}{|D_{n+i-1} D_{n+i}|}$$
$$\leqq \frac{1}{h^2}\left[\frac{1}{(n + h^{-1})(n + 1 + h^{-1})} + \frac{1}{(n + 1 + h^{-1})(n + 2 + h^{-1})} + \cdots \right.$$
$$\left. + \frac{1}{(n + i - 1 + h^{-1})(n + i + h^{-1})}\right].$$

Since the value of the last expression can be made as small as we choose by sufficiently increasing n, the continued fraction will converge uniformly within T'.

The region T' can now be enlarged so as to include as much of the positive half of the w-axis as is desired. The portion of the w-cut between the origin and $w = 4$ was therefore superfluous. In the x-plane we may omit the cut between $x = 1/4k$ and $x = \infty$ which corresponded to this portion.

This completes our determination of the character of the function represented by (1) when $A_1' + A_1'' = 0$. The result reached can be summed up as follows:

THEOREM 4. *If in the continued fraction of Theorem* 1 *the ratios* a_{2n-1}/a_{2n} *and* a_{2n}/a_{2n+1} *for sufficiently large values of n can be expanded into convergent series in powers of* $1/n$ *and if* $A_1' + A_1'' = 0$, *the region within which the continued fraction represents an analytic function can be extended so as to include the entire plane of x with the exception of a cut drawn from* $x = -1/4k$ *to* $x = \infty$

in a direction which forms the continuation of the line joining the origin to $x = -1/4k$.

The only singularities of the analytic function exterior to the cut are poles and the continued fraction converges in the same manner as in Theorem 1.

If $A_1' + A_1'' \neq 0$ *and the real part of* $A_1' + A_1''$ *is positive, the continued fraction represents an analytic function which, in the finite region of the plane, has no other singularities than poles.*

If the real part of $A_1' + A_1''$ *vanishes but the imaginary part does not, then the continued fraction represents an analytic function which at least within a circle described with a radius* $1/4k$ *about the point* $x = 0$ *has no other singularities than poles.*

II. Application of the Fundamental Theorems.

9. (*a*). *The Continued Fraction of Gauss.* The last theorem can be applied directly to the continued fraction of Gauss. For since

$$a_{2n} = -\frac{(\alpha - 1 + n)(\gamma - 1 - \beta + n)}{(\gamma + 2n - 2)(\gamma + 2n - 1)}, \qquad a_{2n+1} = -\frac{(\beta + n)(\gamma - \alpha + n)}{(\gamma + 2n - 1)(\gamma + 2n)},$$

a_n has the limit $k = -1/4$. Moreover,

$$\frac{a_{2n-1}}{a_{2n}} = 1 + \frac{2\beta - 2\alpha + 1}{n} + \frac{1}{n^2}(\quad) + \cdots,$$

$$\frac{a_{2n}}{a_{2n+1}} = 1 - \frac{2\beta - 2\alpha + 1}{n} + \cdots,$$

and $A_1' + A_1'' = 0$. The continued fraction therefore represents a function whose only singularities exterior to the portion of the real axis between $x = 1$ and $x = +\infty$ are poles. It also converges uniformly to the function as its limit in any region bounded by a continuous curve, from which the poles and this portion of the axis are excluded.

It remains to identify this function with the quotient $G(\alpha, \beta, \gamma, x)$ from which Gauss derived the continued fraction. Now

$$G(\alpha, \beta, \gamma, x) = \frac{1}{1+}\,\frac{a_2 x}{1+}\,\frac{a_3 x}{1+}\cdots\frac{a_{2n} x}{+\,1 + a_{2n+1}\,x\,G_1},$$

where

$$G_1 = G(\alpha + n, \beta + n, \gamma + 2n, x).$$

Hence

$$G(\alpha, \beta, \gamma, x) - \frac{N_{2n}}{D_{2n}} = \frac{N_{2n} + a_{2n+1}\, x\, G_1\, N_{2n-1}}{D_{2n} + a_{2n+1}\, x\, G_1\, D_{2n-1}} - \frac{N_{2n}}{D_{2n}}$$

$$= \frac{a_2\, a_3 \cdots a_{2n+1}\, G_1\, x^{2n}}{D_{2n}(D_{2n} + a_{2n+1}\, x\, G_1\, D_{2n-1})}.$$

This shows that when $G(\alpha, \beta, \gamma, x)$ and N_{2n}/D_{2n} are expanded into series in ascending powers of x, their expansions agree to a greater and greater number of terms as n increases. We will prove that $G(\alpha, \beta, \gamma, x)$ coincides with the limit of N_{2n}/D_{2n} by demonstrating the following

THEOREM. *If a sequence of analytic functions converges uniformly to a limit in the vicinity of the origin and their expansions in ascending powers of x, as n increases, agree to a greater and greater number of terms with a given series $P(x)$, this series is the limit of the sequence.**

Let $f_n(x)$ represent the nth of these functions and let

$$f_n(x) = a_0^{(n)} + a_1^{(n)}x + \cdots,$$
$$P(x) = c_0 + c_1x + \cdots.$$

Then, by hypothesis, if the positive integer m is chosen at pleasure, a second integer μ can be so determined that

$$a_0^{(n)} = c_0, \qquad a_1^{(n)} = c_1, \quad \cdots \quad a_{m-1}^{(n)} = c_{m-1}, \qquad n \geqq \mu.$$

Hence, generally,

$$\lim_{n=\infty} a_i^{(n)} = c_i.$$

The theorem asserts that

$$P(x) = \lim_{n=\infty} f_n(x).$$

To prove it, then, we have only to show that the limit of the series

$$a_0^{(n)} + a_1^{(n)}x + \cdots$$

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* This theorem follows from the theorem that, if $s_n(x)$ is a single valued, analytic function of x throughout the two-dimensional region T of the complex plane for $n = 1, 2, \ldots$ and if $s_n(x)$ converges uniformly in T when $n = \infty$, its limit being denoted by $F(x)$; then $F^{(i)}(x) = \lim_{n=\infty} s_n^{(i)}(x)$, $i = 1, 2, 3, \ldots$, and the proof is now readily completed. *Cf.* de la Vallée-Poussin, *Annales de la Société scientifique de Bruxelles*, seconde partie, Mémoires, vol. 17, 1893, p. 324. [ED.]

is equal to the series of the limits:

$$\lim_{n=\infty} a_0^{(n)} + \lim_{n=\infty} a_1^{(n)} x + \cdots\cdots .$$

This follows immediately from the uniform convergence. Thus the theorem is proven, and the limit of the continued fraction of Gauss is the function $G(\alpha, \beta, \gamma, x)$.

If α, β, γ are real, an upper limit for the number of poles of this function exterior to the positive half of the real axis can be obtained with the aid of Theorem 3. It will be seen at once that it can not be greater than $2N$, if N is the smallest positive integral value of n for which the following inequalities simultaneously hold:

$$\alpha + n - 1 > 0, \qquad \beta + n > 0$$

$$\gamma - 1 - \beta + n > 0, \qquad \gamma - \alpha + n > 0, \qquad \gamma + 2\,n > 0.$$

This furnishes also an upper limit for the number of zeros of $F(\alpha, \beta, \gamma, x)$ and its analytic continuation which lie without the positive half-axis.

10. (*b*). *The Continued Fraction for Bessel's Quotient J_n/J_{n-1}.* From the well known equation connecting three consecutive related Bessel's functions,

$$\frac{J_{n-1}(x)}{J_n(x)} = \frac{2n}{x} - \frac{J_{n+1}(x)}{J_n(x)},$$

the continued fraction

$$\frac{2n}{x} - \cfrac{1}{\cfrac{2n+2}{x} - \cfrac{1}{\cfrac{2n+4}{x} - \cdots}}$$

is derived. This may be transformed so as to become

$$\frac{2n}{x} - \frac{\frac{x}{2n+2}}{1} - \frac{\frac{x^2}{(2n+2)(2n+4)}}{1} - \frac{\frac{x^2}{(2n+4)(2n+6)}}{1} - \cdots \tag{18}$$

Also by removing a factor x^{-1} and placing $z = -x^{-2}$, it may be thrown into the form

$$2n + \frac{1}{(2n+2)z +} \; \frac{1}{(2n+4) +} \; \frac{1}{(2n+6)z +} \cdots \tag{19}$$

The application of Theorem 1 to the continued fraction (18) shows that it represents a function which is meromorphic over the entire plane. The identity of the function with J_{n-1}/J_n can be established easily with the aid of the theorem of §9.

When n is real, Theorems 2 and 3 can both be applied to (19). We thus find that the number of poles of the function defined by (19) which lie without the negative half of the axis is equal to zero if $n > -1$. When n lies between the two negative integers, $-m$ and $-(m+1)$, the integer m furnishes an upper limit for the number of such poles. Now to each of these poles there corresponds a pair of imaginary poles of the meromorphic function defined by (18). Since the poles of the latter function are the roots of J_n, we conclude that all the roots of J_n are real if $n > -1$. If $n < -1$, the number of pairs of conjugate imaginary roots can not exceed m. As a matter of fact, this is the exact number.*

11. (c). *Heine's Continued Fraction.* One of the generalizations of the hypergeometric series $F(\alpha, \beta, \gamma, x)$, which was proposed by Heine† is the series

$$\phi(\alpha, \beta, \gamma, q, x) = 1 + \frac{(1-q^\alpha)(1-q^\beta)}{(1-q)(1-q^\gamma)} + \frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q)(1-q^2)(1-q^\gamma)(1-q^{\gamma+1})} + \cdots.$$

This satisfies a difference equation of the second order. The hypergeometric series is the limiting case obtained by letting q approach unity. Without in any way restricting the generality of the above series, we may assume q to be less than unity.

For the quotient

$$\frac{\phi(\alpha, \beta+1, \gamma+1, q, x)}{\phi(\alpha, \beta, \gamma, q, x)}$$

Heine obtained a continued fraction of the form (1) in which

$$a_{2n} = q^{\beta+n-1}\frac{(1-q^{\alpha+n-1})(1-q^{\gamma-\beta+n-1})}{(1-q^{\gamma+2n-2})(1-q^{\gamma+2n-1})},$$

$$a_{2n+1} = q^{\alpha+n-1}\frac{(1-q^{\beta+n})(1-q^{\gamma-\alpha+n})}{(1-q^{\gamma+2n-1})(1-q^{\gamma+2n})}.$$

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* *Math. Annalen*, vol. 33, 1888, p. 258.

† *Crelle*, vol. 34, 1847.

When n increases indefinitely, the limit of both of these coefficients is obviously zero. Theorem 1 is therefore applicable and shows that the continued fraction represents a meromorphic function, which, by the theorem of §9, may be proved to be identical with the above quotient.

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ON THE DIFFERENTIATION OF AN INFINITE SERIES TERM BY TERM.

BY M. B. PORTER.

IF

$$f(x) = \sum_1^\infty u_n(x), \qquad a < x < b,$$

denote a convergent series of single valued, differentiable, real functions of the real variable x, the ordinary sufficient condition that the derivative $f'(x)$ exist and that

$$f'(x) = \sum_1^\infty u_n'(x),$$

namely, that each term $u_n(x)$ be differentiable and that the series $\sum_1^\infty u_n'(x)$ be uniformly convergent, is practically limited to the case where the uniformity can be tested by means of a so called M-series. That is, we shall have

$$f'(x) = \sum_1^\infty u_n'(x), \qquad a < x < b,$$

if there exist a convergent series of positive terms, $\sum_1^\infty M_n$, such that

$$|u_n'(x)| \leqq M_n, \qquad a < x < b. \tag{A}$$

The practical importance of this test in that, in particular, it is immediately applicable to power series and thus leads to a simple as well as rigorous treatment of undetermined coefficients, renders an elementary proof of it desirable. The following proof, while probably not new,* has not come to the notice of the writer elsewhere, and seems to be sufficiently elementary.

If x and $x + \Delta x$ lie in the interval of convergence of the series

$$f(x) = \sum_1^\infty u_n(x),$$

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* Dini, *Functionen einer reellen Grösse*, p. 154, uses the Mean Value Theorem, but as he deduces a more general criterion, his proof is much less elementary.

then, by the primary definition of convergence, we have

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\sum_{1}^{\infty}\frac{u_n(x+\Delta x)-u_n(x)}{\Delta x},$$

and, by the first Mean Value Theorem,

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\sum_{1}^{\infty}u_n'(x+\theta_n\Delta x),\qquad 0<\theta_n<1. \tag{1}$$

If ϵ be an arbitrarily assigned positive number as small as we please, (A) tells us that a positive integer N can be found such that

$$\left|\sum_{N}^{\infty}u_n'(x+\theta_n\Delta x)\right|<\frac{\epsilon}{2}. \tag{2}$$

Furthermore, since each term $u_n(x)$ has a derivative $u_n'(x)$, a number δ can be found such that, if $|\Delta x|<\delta$,

$$\frac{u_n(x+\Delta x)-u_n(x)}{\Delta x}=u_n'(x+\theta_n\Delta x)$$

will differ from its limit, $u_n'(x)$, numerically by less than $\epsilon/2n$, so that

$$\left|\sum_{1}^{n}u_n'(x+\theta_n\Delta x)-\sum_{1}^{n}u_n'(x)\right|<\frac{\epsilon}{2}. \tag{3}$$

Hence it follows that

$$\left|\frac{f(x+\Delta x)-f(x)}{\Delta x}-\sum_{1}^{\infty}u_n'(x)\right|<\epsilon,\qquad |\Delta x|<\delta,$$

which proves the theorem.

The continuity, or even the integrability, of the derivatives $u_n'(x)$ is not postulated in the above proof, since the Mean Value Theorem merely requires that the derivatives exist.

The same proof can be applied when x denotes the complex variable, x_1+x_2i, the Mean Value Theorem* for functions of a complex variable now being employed.

NEW HAVEN, CONNECTICUT,
AUGUST, 1901.

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* Stolz, *Diff. u. Integralrechnung*, vol. 2, p. 95.

A NOTE ON GEODESIC CIRCLES.

By J. K. Whittemore.

Bianchi defines a geodesic circle as the locus of a point on a surface at a constant geodesic distance from a fixed point of the surface.[*] In this note I use the term in this sense. Darboux calls a curve of constant geodesic curvature a geodesic circle.[†] Bianchi states in a footnote[‡] that all the geodesic circles of a surface are curves of constant geodesic curvature, when the total curvature of the surface is constant.

In this note I shall prove a set of more general theorems from which Bianchi's statement and its converse follow at once. I use the term "geodesic circle" in Bianchi's sense. My theorems are the following:

Theorem 1. *If, on a surface, there exists a family of concentric geodesic circles such that the geodesic curvature of each curve of the family is constant, then the total curvature of the surface is constant along each curve of the family, and the surface is applicable to a surface of revolution so that the geodesic circles fall on the circles of latitude of this surface.*

Theorem 2. *Conversely, if, on a surface, there exists a family of concentric geodesic circles such that the total curvature of the surface is constant along each curve of the family, then the geodesic curvature of each geodesic circle is constant, and the surface is applicable to a surface of revolution by the former theorem.*

Theorem 3. *Finally, if a surface is applicable to a surface of revolution so that the members of a family of concentric geodesic circles of the surface fall upon the circles of latitude of the surface, then the geodesic circles are curves of constant geodesic curvature and of constant total curvature.*

I suppose the common centre of the geodesic circles to be an ordinary point of the surface. I choose as curvilinear coordinates on the surface, the geodesic distance, u, from the common centre of the circles, and the angle, v, which a geodesic makes at this point with some fixed direction on the surface at this point. Then the linear element of the surface has the form,

$$ds^2 = du^2 + C^2\,dv^2,$$

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[*] Bianchi. *Vorlesungen über Differentialgeometrie*, p. 160.

[†] Darboux. *Théorie générale des surfaces*, vol. 3, p. 151.

[‡] *l. c.*, p. 162.

where
$$C_{\substack{u=0\\ v=v}} = 0, \qquad \left(\frac{\partial C}{\partial u}\right)_{\substack{u=0\\ v=v}} = 1.$$

The geodesic curvature of one of the geodesic circles is given by the formula:

$$\frac{1}{\rho_u} = -\frac{1}{C}\frac{\partial C}{\partial u}.$$

The total curvature of the surface at any point is*

$$\kappa = -\frac{1}{C}\frac{\partial^2 C}{\partial u^2}$$

Consider first Theorem 3. The element of arc dS of a surface of revolution can be expressed in the form

$$dS^2 = d\sigma^2 + \phi(\sigma)^2 d\theta^2,$$

where $d\sigma$ denotes the element of arc of a meridian curve, $y = f(x)$, and $\phi(\sigma) = y$; the longitude being measured by θ. According to the hypothesis of the theorem, it is possible to determine u and v as functions of σ and θ such that, when $u =$ const., $\sigma =$ const. also and furthermore that $dS^2 = ds^2$, or

$$du^2 + C^2 dv^2 = d\sigma^2 + \phi^2 d\theta^2. \tag{1}$$

Let $u = u(\sigma, \theta)$, $v = v(\sigma, \theta)$. Then, from the first condition it follows that

$$\frac{\partial u}{\partial \theta} = 0.$$

Substituting for du and dv their values in terms of $d\sigma$ and $d\theta$, we show further that

$$\frac{\partial u}{\partial \sigma} = 1, \qquad \frac{\partial v}{\partial \sigma} = 0.$$

Hence $u = \sigma$ and v is a function of θ alone. Since the assignment of the individual geodesics $v = v_1$ to the individual meridians $\theta = \theta_1$ is still arbitrary, we may make this assignment in such a way that $v = \theta$. It then follows from (1) that

$$C^2 = \phi(\sigma)^2,$$

i. e., that C is a function of u alone. The same will be true of $1/\rho_u$ and κ: hence the theorem.

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* For these formulas see Bianchi, l. c., pp. [illegible].

Let us now suppose, to prove Theorem 1, that the geodesic circles, $u =$ constant, are curves of constant geodesic curvature. Then

$$\frac{1}{\rho_u} = - \frac{1}{C} \frac{\partial C}{\partial u} = - U'$$

where U is a function of u alone. We have by integration

$$C = e^{U+V}$$

where V is a function of v alone.

Now, since $\left(\frac{\partial C}{\partial u}\right)_{u=0} = 1$, it follows that

$$1 = U'(0)e^{U(0)+V(v)},$$

and hence that V is a constant. C is, therefore, a function of u alone, and Theorem 1 is proved.*

Finally, to prove Theorem 2, suppose that the geodesic circles, $u =$ constant, are for the surface curves of constant total curvature. Then is

$$\kappa = - \frac{1}{C} \frac{\partial^2 C}{\partial u^2} = - F(u) \tag{2}$$

where $F(0)$ is finite, since the centre of the geodesic circles is an ordinary point of the surface. In order to integrate equation (2), let U be a solution of the ordinary differential equation

$$U'' + U'^2 = F(u)$$

which vanishes for $u = 0$. Then we have

$$\frac{1}{C} \frac{\partial^2 C}{\partial u^2} = U'' + U'^2$$

Next, replace C by z where

$$C = z e^U;$$

z is an unknown function of u and v, which must however vanish for $u = 0$. For the determination of z we have the equation:

$$\frac{\partial^2 z}{\partial u^2} + 2U' \frac{\partial z}{\partial u} = 0.$$

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* For a proof of the theorem that, if the element of arc of a surface can be written in the form $dS^2 = Edu^2 + Gdv^2$, where E, G are functions of u alone, the surface is applicable to a surface of revolution, *cf.* Picard, *Traité d'analyse*, vol. 1, p. 423.

Integrating this equation, the general value of z is found to be

$$z = e^{V} \int_0^u e^{-2U}\, du + V_1,$$

where V and V_1 are arbitrary functions of v. Since now z and u vanish together, V_1 is identically zero, and

$$C = e^{U+V} \int_0^u e^{-2U}\, du.$$

Furthermore, since for $u = 0$, we have

$$\frac{\partial C}{\partial u} = 1, \qquad U = 0,$$

it follows that for all values of v

$$e^{V} = 1.$$

Hence V is identically zero, and

$$C = e^{u} \int_0^u e^{-2U}\, du$$

and is a function of u alone. Theorem 2 follows at once.[*]

HARVARD UNIVERSITY,
CAMBRIDGE, MASSACHUSETTS, MAY 18, 1901.

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[*] Picard, *l. c.*

NOTE ON THE FUNCTIONS DEFINED BY INFINITE SERIES WHOSE TERMS ARE ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE; WITH CORRESPONDING THEOREMS FOR DEFINITE INTEGRALS.

BY W. F. OSGOOD.

LET

$$f_1(x) + f_2(x) + \cdot \cdot \cdot \qquad (a)$$

be an infinite series of real functions of the real variable x, $f_n(x)$ being single valued and continuous throughout the interval (a, b) : $a \leqq x \leqq b$, and let the series converge for all values of x pertaining to this interval. Its value shall be denoted by $F(x)$. The class of functions $F(x)$ thus defined and the mode of convergence of the series have been studied at length.* In particular, the function $F(x)$ need not be continuous throughout any interval lying in (a, b). This class is coextensive with the class which, at first sight, might appear to be smaller and which is obtained by imposing on $f_n(x)$ the further restriction that it be *analytic* throughout the interval (a, b). For, let

$$s_n(x) = f_1(x) + \cdot \cdot \cdot \cdot + f_n(x).$$

Then there exists a function $\sigma_n(x)$ analytic throughout the interval (a, b) and differing numerically from $s_n(x)$ uniformly by an arbitrarily small amount, *i. e.* $\sigma_n(x)$ satisfies the relation

$$|s_n(x) - \sigma_n(x)| < \epsilon,$$

where ϵ is an arbitrarily small positive quantity.† It is readily seen that the function $\sigma_n(x)$ converges towards the value $F(x)$; for

$$|F(x) - s_n(x)| < \epsilon', \qquad \left.\begin{array}{l}\epsilon', \text{ arbitrary;} \\ n, \text{ suitably chosen}\end{array}\right\};$$

$$|s_n(x) - \sigma_n(x)| < \epsilon ;$$

$$\therefore \qquad |F(x) - \sigma_n(x)| < \epsilon + \epsilon'.$$

Now form the series :

---

* *Cf.* Schoenflies, Bericht über die Mengenlehre, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 8, 1900, chap. 7, p. 217.

† $\sigma_n(x)$ may, in fact, be chosen as a polynomial. This follows from Weierstrass's theorem that any continuous function of a real variable can be represented by a uniformly convergent series of polynomials. *Cf.* Picard, *Traité d'analyse*, vol. 1, 1st ed., p. 258.

$$\phi_1(x) + \phi_2(x) + \cdot \cdot \cdot \cdot ,$$

where $\phi_n(x) = \sigma_n(x) - \sigma_{n-1}(x)$, $n > 1$; $\phi_1(x) = \sigma_1(x)$. Then this is the desired series; namely, a series of analytic functions having the same function $F(x)$ for its value as the original series.

If we pass from the domain of real to that of complex variables and functions, the precise analogon would be a series of complex functions, single valued and continuous throughout a two-dimensional region T of the complex z-plane and convergent at each point of this region. Such series are without interest unless their terms are analytic functions of z. We shall here consider *the class of functions*

$$F(z) = f_1(z) + f_2(z) + \cdot \cdot \cdot \cdot$$

defined by a series of functions, each of which is analytic throughout a two-dimensional region T of the complex z-plane, the series being assumed to converge for every point z lying within this region.

The principal results of the paper are stated in Theorems I and II, and are of a very general nature. The condition of uniform convergence is nowhere imposed. In fact, Theorem II presupposes only the bare convergence of the series — nothing more. The proof of Theorem II depends on a theorem in the theory of sets of points, which is quoted in No. 5, and which has already proved of value in the study of general questions in the Theory of Functions.

We observe that from Theorem II follows at once a proof of the theorem that *a function of z which is continuous throughout T cannot be represented by means of a series of functions each analytic throughout T unless it is at least in parts of T analytic.*

§1.

1. Theorem I. *If the infinite series*

$$f_1(z) + f_2(z) + \cdot \cdot \cdot \cdot , \qquad (1)$$

whose terms are all functions of z, single valued and analytic throughout a region T of the complex z-plane, converges for all values of z pertaining to a set of points which is everywhere dense throughout T (and which, in particular, may be enumerable); and if, furthermore, the relation

$$| f_1(z) + \cdot \cdot \cdot + f_n(z) | \leqq G$$

holds for all points z of this set (and hence of T) and for all values of n, G

being a positive constant, then the series converges for all values of z in T and the function $F(z)$ defined by the series:

$$F(z) = f_1(z) + f_2(z) + \cdot \cdot \cdot \cdot$$

is analytic throughout T.

To prove the theorem, we set

$$f_n(z) = u_n(x, y) + iv_n(x, y), \qquad z = x + iy,$$

form the series of harmonic functions

$$u_1(x, y) + u_2(x, y) + \cdot \cdot \cdot , \tag{2}$$

and show that this series converges uniformly throughout the neighborhood of every interior point A of T. Hence the function $u(x, y)$ defined by (2) is harmonic in T and the series can be differentiated term by term, the resulting series converging likewise uniformly. The series of conjugate functions,

$$v_1(x, y) + v_2(x, y) + \cdot \cdot \cdot \tag{3}$$

converges for each interior point of T and the function $v(x, y)$ that it defines is conjugate to the function $u(x, y)$.

2. The details of the proof are as follows. We begin by expressing certain harmonic functions with which we have to deal by means of Poisson's Integral.* About the point A describe a circle lying wholly within T. Denote its radius by R, any interior point (x, y) by P, and any point on the circumference by Q. Further, denote the angles that AP and AQ make with the positive axis of x by ϕ and ψ respectively, and the distance AP by r. Then, if $w(x, y)$ is any function, continuous throughout the region composed of the interior and the circumference of the circle and harmonic at all points within the circle, and if the values of $w(x, y)$ along the circumference of the circle are denoted by $W(\psi)$, Poisson's Integral gives the following representation of $w(x, y)$ at the interior points of the circle:

$$w(x, y) = \frac{1}{2\pi} \int_0^{2\pi} W(\psi) \frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2} \, d\psi. \tag{4}$$

Consider the function

$$s_n(x, y) = u_1(x, y) + \cdot \cdot \cdot + u_n(x, y).$$

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[*] *Cf.*, for example, Picard's *Traité d'analyse*, vol. 2, p. 16.

Its partial derivatives, $\frac{\partial s_n}{\partial x}$ *and* $\frac{\partial s_n}{\partial y}$, *are finite throughout the neighborhood of the point A.* We will choose as this neighborhood, $\mathfrak{R}_A$, the interior of a circle with A as centre and radius $\frac{1}{2}R$. From (4),

$$s_n(x, y) = \frac{1}{2\pi}\int_0^{2\pi} S_n \frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2}\, d\psi;$$

hence

$$\frac{\partial s_n}{\partial x} = \frac{1}{2\pi}\int_0^{2\pi} S_n \frac{\partial}{\partial x}\frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2}\, d\psi,$$

with a similar formula for $\partial s_n/\partial y$. Differentiation under the sign of integration is here allowable.[*] Denote by K a quantity that is at least as large as the maximum numerical value of either of the expressions

$$\frac{\partial}{\partial x}\frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2}, \qquad \frac{\partial}{\partial y}\frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \phi) + r^2},$$

(x, y) being any point of $\mathfrak{R}_A$ and $0 \leqq \psi < 2\pi$. Then, since $|S_n| \leqq G$ for all values of ψ, it follows that

$$\left|\frac{\partial s_n}{\partial x}\right| \leqq KG, \qquad \left|\frac{\partial s_n}{\partial y}\right| \leqq KG, \qquad q.\ e.\ d.$$

3. We are now in a position to prove that the function $s_n(x, y)$ converges uniformly toward a limit throughout the region $\mathfrak{R}_A$. The proof follows at once from a theorem which has been stated and proved for functions of a single variable by de la Vallée-Poussin.[†] The extension to functions of n variables is immediate. The theorem is as follows.

Let $s_n(x)$ *be a function of* x, *continuous throughout the interval* $a \leqq x \leqq b$ *and having a finite derivative*[‡] *for all values of* x *in the interval and for* $n = 1, 2 \ldots$; *i. e.*

$$|s'_n(x)| < L$$

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[*] *Cf.* Picard, *Traité d'analyse*, vol. 1, p. 29.

[†] *Annales de la Société scientifique de Bruxelles*, première partie, vol. 17, 1893, p. 8.

[‡] Instead of a finite derivative it is enough to require that the difference-quotient remain finite, *i. e.* that

$$|s_n(x + h) - s_n(x)| \leqq Nh,$$

where x and $x + h$ are any two points of the interval, $n = 1, 2, \ldots$, and N is a constant.

for all such values of x and n, L being a constant. Furthermore, let $s_n(x)$ converge toward a limit for each one of a set of values of x everywhere dense throughout this interval, when $n = \infty$. Then $s_n(x)$ converges uniformly toward a limit for the totality of values of x pertaining to the interval.

Thus the function $s_n(x, y)$ is seen to converge uniformly toward a limit, $u(x, y)$, throughout the region consisting of $\mathfrak{R}_A$ and its boundary. $s_n(x, y)$ is a harmonic function throughout $\mathfrak{R}_A$ and it takes on a continuous set of boundary values which, when $n = \infty$, converge uniformly toward a limit. The limit of $s_n(x, y)$ is, therefore, harmonic throughout $\mathfrak{R}_A$* and

$$\frac{\partial u}{\partial x} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} + \cdots\cdots .$$

This series converges uniformly. The corresponding theorems are true for the partial derivatives of $s_n(x, y)$ and $u(x, y)$ with respect to y.

4. It remains to show that the series (3) converges and defines a function conjugate to $u(x, y)$. Consider the series

$$\sum_{n=1}^{\infty} \int_{(x_0, y_0)}^{(x, y)} \frac{\partial u_n}{\partial x} dy - \frac{\partial u_n}{\partial y} dx. \tag{5}$$

Since the series

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} \frac{\partial u_n}{\partial x}, \qquad \frac{\partial u}{\partial y} = \sum_{n=1}^{\infty} \frac{\partial u_n}{\partial y}$$

converge uniformly, these series can be integrated term by term and the value of the expression (5) is

$$\bar{v}(x, y) = \int_{(x_0, y_0)}^{(x, y)} \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx,$$

which is a function conjugate to $u(x, y)$. On the other hand, the value of the general term of (5) is $v_n(x, y) - v_n(x_0, y_0)$. Hence, the series

$$\sum_{n=1}^{\infty} v_n(x, y) - v_n(x_0, y_0)$$

converges and, if (x_0, y_0) is chosen as one of the points for which the series

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* This follows from a theorem of Harnack's. *Cf.* Harnack, *Math. Annalen*, vol. 35, 1889, p. 22, or Picard, *Traité d'analyse*, vol. 2, p. 56. That such a series can be differentiated term by term and that the resulting series is uniformly convergent can readily be shown by means of the representation by Poisson's Integral.

(1) and hence the series $\sum_{n=1}^{\infty} v_n(x_0, y_0)$ converges, it is seen that the series (3) converges. If $v(x, y)$ denotes its value, then

$$v(x, y) = \overline{v}(x, y) + v(x_0, y_0),$$

and the function $v(x, y)$ is conjugate to $u(x, y)$,

This completes the proof of Theorem I.

§2.

5. THEOREM II. *If the infinite series*

$$f_1(z) + f_2(z) + \cdot \cdot \cdot \cdot ,$$

whose terms are all functions of z, single valued and analytic throughout a region T of the complex z-plane, converges for all values of z lying within this region, then there exists one or more analytic functions $F_1(z)$, $F_2(z)$, . . . such that the domain of definition of $F_i(z)$ coincides, in part or wholly, with a region t_i lying in T and, for all values of z in t_i,

$$F_i(z) = F(z),$$

where $F(z)$ denotes the value of the series. Every interior point of T either lies within some t_i or else a part of one or more regions t_i extends into the neighborhood of this point.[*]

The functions $F_i(z)$ may be infinite in number. They are always enumerable.

Let t be an arbitrary, two-dimensional region lying wholly within T. It may happen that, for a suitably chosen (large) positive quantity M, the absolute value of

$$\mathfrak{S}_n(z) = f_1(z) + \cdot \cdot \cdot \cdot f_n(z)$$

is less than M for all points z of t and for all values of n. In that case, it follows from Theorem I that the series defines a function analytic throughout t.

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[*] This is not the same as saying that the point lies on the boundary of one or more regions t_i. This case is included in the statement. But it may happen that the point lies on the boundary of no region t_i, but is a point of condensation of boundaries of an infinite number of regions t_i.

It may be objected that the notation $F_i(x)$, $i = 1, 2, \ldots$, assumes from the start that these functions are enumerable. No use, however, is made of this assumption in the proof, and so it is a matter of no moment. An adequate notation could be devised by allowing i to take on other than integral values, since the number of the functions $F_i(z)$ obviously does not exceed the power (Mächtigkeit) of the continuum.

In any case, I say that *it is always possible so to choose M that, throughout some two-dimensional region t' lying in t, the relation holds:*

$$|\mathfrak{S}_n(z)| \leqq M$$

for all values of z in t' and for $n = 1, 2, \ldots$, and hence $F(z)$ is analytic throughout t', $F(z)$ denoting the value of the series.

Suppose this statement were not true. Choose a set of positive quantities M_i increasing indefinitely with i. Denote the points of t at which $|\mathfrak{S}_n(z)| \leqq M_i$ for all values of n by P_i. Then the assumption is that the points of P_i nowhere in t form a two-dimensional continuum. They cannot, then, even be everywhere dense throughout any such continuum, since $|\mathfrak{S}_n(z)|$ is a continuous function of z. The points of P_i are all found in P_{i+1}. Denote by P the set of points, each of which ultimately appears in some P_i:

$$P = \lim_{i=\infty} P_i.$$

Now each point of t ultimately appears in some P_i; hence P is identical with the totality of points of t. But t is a two-dimensional continuum. We are thus led to a contradiction of a theorem of the theory of sets of points, which says:[*]

In a two-dimensional region t of a plane let sets of points P_1, P_2, . . . be given, which have the following properties:

1) *the points of P_i are all contained among the points of P_{i+1};*

2) *in no two-dimensional region whatever are the points of P_i everywhere dense.*

Furthermore, let

$$P = \lim_{i=\infty} P_i$$

be the totality of points that participate in the sets P_i. Then no part of P can form a two-dimensional continuum.

6. To complete the proof of the theorem, let A be any interior point of T. If A lies in a region t_i, the theorem is granted. If not, consider an arbitrary neighborhood t of A. Then there exists one or more regions t_i having points in common with t. But t is *any* neighborhood of A.

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[*] This theorem (with an unessential restriction) I obtained for the case of one-dimensional regions in an article on Non-Uniform Convergence and the Integration of Series Term by Term, *Am. Jour. of Math.*, vol. 19, 1897, p. 173. The proof there given is indirect. For a simple, direct proof *cf. Math. Annalen*, vol. 53, 1900, p. 462.

Finally, the regions t_i are enumerable. This is a theorem of the theory of sets that has long since been proved. In fact, the internal area[*] of each t_i is a positive quantity I_i, and the sum of any number of these quantities does not exceed the internal area of T itself. If, then, we choose a set of steadily decreasing quantities $\epsilon_1, \epsilon_2, \ldots$ with $\lim_{n=\infty} \epsilon_n = 0$, there will be but a finite number of t_i's for each of which $I_i > \epsilon_1$. These we write in the first row of a table and arrange at pleasure in that row. Then there will be but a finite number of further t_i's for which $I_i > \epsilon_2$. These we write in the second row of the table and arrange at pleasure in that row. And so on. Each t_i must ultimately appear in some row of the table, and this proves the theorem.

7. The question arises as to whether the number of the regions t_i can ever actually be infinite. The answer is in the affirmative, as can readily be shown by means of methods due to Runge.[†] Thus, for example, a series can be constructed satisfying all the conditions of the theorem for the interior of the square bounded by the axes of reals and pure imaginaries and by the lines $x = 1$ and $y = 1$, whose value $F(z)$ is $\frac{1}{2}$ when $x \geqq \frac{1}{2}$; $\frac{1}{3}$, when $x \geqq \frac{1}{3}$ but $< \frac{1}{2}$; $\frac{1}{4}$, when $x \geqq \frac{1}{4}$ but $< \frac{1}{3}$; etc.

Two or more of the functions $F_i(z)$ may be capable of being deduced from one another by means of analytic continuation, so that they form parts of one and the same analytic function. This, again, can be demonstrated by Runge's methods. Thus one can, for example, construct a series convergent for the square just considered and having the value $\frac{1}{2}$ when $x \neq \frac{1}{2}$, the value 1 when $x = \frac{1}{2}$. In fact, we may draw any enumerable set of lines whatever in T, each of which has its extremities in two distinct points of the boundary of T, the lines not cutting themselves or each other or clustering about any point lying within T.[‡] Then we may assume arbitrary values along these lines, provided merely that the values for a given line are such as an analytic function could take on along that line. To each of the regions t_i into which T is thus divided we may assign an arbitrary function $F_i(z)$ analytic throughout this region. Then we can construct a series whose value $F(z)$ shall correspond to the values and the functions thus assumed.

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[*] For the definition of the *internal area* of a set of points *cf.*, for example, Jordan, *Cours d'analyse*, vol. 1, 2d ed., 1893, §36.

[†] Runge, *Acta Math.*, vol. 6, 1885, p. 229.

[‡] Examples in which these conditions are not fulfilled can also be readily constructed.

§3

8. By means of Theorem I, a theorem of Stieltjes may be generalized. Stieltjes proved in substance the following theorem:*

If

$$f_1(z) + f_2(z) + \cdot \cdot \cdot \cdot$$

is a series of functions, each single valued and analytic throughout a region T of the z-plane, and if

$$|f_1(z) + \cdot \cdot \cdot \cdot + f_n(z)| < G$$

for all values z of T and for all values of n, G being a constant; if, furthermore, the above series converges uniformly throughout some two-dimensional region t lying in T, then the series converges throughout the entire region T and its value is a function $F(z)$ which is analytic throughout T.

From Theorem I it follows that the requirement of *uniform convergence throughout t* can be replaced by the less restrictive one that the series *converge for each point of a set everywhere dense throughout t.* Such a set may always be so assumed as to be enumerable.

§4.

9. Theorems I and II were stated for infinite series. Corresponding theorems may be stated for definite integrals and the proofs will hold without modification.

Let $\phi(a, z)$ be a (complex) function of the real variable a in the interval A: $a_0 \leqq a \leqq a_1$ and of the complex variable z in a region T of the z-plane. For each value of a in A, $\phi(a, z)$ shall be an analytic function of z throughout T; and for each value of z in T, $\phi(a, z)$ shall be a continuous function of a throughout A.† Then the integral

$$\int_{a_0}^{a_1} \phi(a, z)da$$

will converge for each value of z in T. Since its value is invariant of the mode of division of the interval A, we may choose a special mode depending on a parameter n that passes through only positive integral values; for example, we may divide the interval into n equal parts. Let this be done and set

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* Stieltjes, *Annales de la Faculté des Sciences de Toulouse*, vol. 8, 1894, p. J. 56.

† This condition may be replaced by the broader one that $\phi(a, z)$ be an integrable function of a throughout A.

$$s_n(z) = \sum_{i=1}^{n} \phi\,(a_0 + i\,\Delta a,\ z)\,\Delta a, \qquad \Delta a = \frac{a_1 - a_0}{n}.$$

Then $s_n(z)$ is a function of z analytic throughout the region T for each positive integral value of n. If it satisfies the further conditions of Theorem I or II respectively, then the same conclusions will hold regarding the function

$$F(z) = \int_{a_0}^{a_1} \phi(a,\ z)\,da$$

as in the former case for the function $F(z)$ defined by the series.

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NIM, A GAME WITH A COMPLETE MATHEMATICAL THEORY.

By Charles L. Bouton.

The game here discussed has interested the writer on account of its seeming complexity, and its extremely simple and complete mathematical theory.* The writer has not been able to discover much concerning its history, although certain forms of it seem to be played at a number of American colleges, and at some of the American fairs. It has been called Fan-Tan, but as it is not the Chinese game of that name, the name in the title is proposed for it.

1. Description of the Game. The game is played by two players, A and B. Upon a table are placed three piles of objects of any kind, let us say counters. The number in each pile is quite arbitrary, except that it is well to agree that no two piles shall be equal at the beginning. A play is made as follows:—The player selects one of the piles, and from it takes as many counters as he chooses; one, two, . . ., or the whole pile. The only essential things about a play are that the counters shall be taken from a single pile, and that at least one shall be taken. The players play alternately, and the player who takes up the last counter or counters from the table wins.

It is the writer's purpose to prove that if one of the players, say A, can leave one of a certain set of numbers upon the table, and after that plays without mistake, the other player, B, *cannot* win. Such a set of numbers will be called a *safe combination*. In outline the proof consists in showing that if A leaves a safe combination on the table, B at his next move cannot leave a safe combination, and whatever B may draw, A at his next move can again leave a safe combination. The piles are then reduced, A always leaving a safe combination, and B never doing so, and A must eventually take the last counter (or counters).

2. Its Theory. A *safe combination* is determined as follows: Write the number of the counters in each pile in the binary scale of notation,† and

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* The modification of the game given in §6 was described to the writer by Mr. Paul E. More in October, 1899. Mr. More at the same time gave a method of play which, although expressed in a different form, is really the same as that used here, but he could give no proof of his rule.

† For example, the number 9, written in this notation, will appear as

$$1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 1001.$$

place these numbers in three horizontal lines so that the units are in the same vertical column. If then the sum of *each* column is 2 or 0 (*i. e.* congruent to 0, mod. 2), the set of numbers forms a safe combination. For example,

$$\begin{array}{r} 1\ 0\ 0\ 1, \\ 1\ 0\ 1, \\ 1\ 1\ 0\ 0, \end{array}$$

or 9, 5, 12 is a safe combination. It is seen at once that if any two numbers be given, a third is always uniquely determined which forms a safe combination with the two given numbers. Moreover, it is obvious that if a, b, c form a safe combination any two of the numbers determine the remaining one, that is, the system is closed. A particular safe combination which is used later is that in which two piles are equal and the third is zero. In the proofs which follow, the binary scale of notation is used throughout.

THEOREM I. *If A leaves a safe combination on the table, B cannot leave a safe combination on the table at his next move.* B can change only *one* pile, and he must change one. Since when the numbers in two of the piles are given the third is uniquely determined, and since A left the number so determined in the third pile (*i. e.*, the pile from which B draws) B cannot leave that number. Hence B cannot leave a safe combination.

THEOREM II. *If A leaves a safe combination on the table, and B diminishes one of the piles, A can always diminish one of the two remaining piles, and leave a safe combination.* Consider first an example. Suppose A leaves the safe combination *nine*, *five*, *twelve*, and that B draws *two* from the first pile, leaving the numbers *seven*, *five*, *twelve*, or

$$\begin{array}{r} 1\ 1\ 1, \\ 1\ 0\ 1, \\ 1\ 1\ 0\ 0. \end{array}$$

If A is to leave a safe combination by *diminishing* one of the piles, it is clear that he must select the third pile, that containing *twelve*. The number which is safe with 111 and 101 is 10, or *two*. Hence A must leave *two* in the pile which contains *twelve*, or draw *ten* from that pile, and by doing so he leaves a safe combination.

To prove the general theorem, let the numbers, expressed in the binary scale, be written with the units in a vertical column, and suppose that A left a safe combination. B selects one of the piles and diminishes it. When a number of the binary scale is diminished it is essential to notice that in going over the number from left to right the first change which occurs is that some 1 is changed to 0, for if a 0 were changed to 1 the number would be increased whatever changes were made in the subsequent digits.[*] Consider, then, this first column, counting from the left, in which a change occurs. One and only one of the other two numbers will contain 1 in this same column, for *A* left a safe combination. Let *A* select the pile which contains the 1 in this column, and change the number by writing 0 in this column, and filling the remaining columns to the right with 0 or 1 so as to make a safe combination. The columns to the left remain unchanged, since they already have the required form. The new number so formed will be *less* than that in the pile which *A* selected. Hence whatever *B* draws, *A* can always diminish one of the piles, and leave a safe combination. That is, if *A* at any play can leave a safe combination on the table, he can do so at every subsequent play, and *B* never can do so.

If the play continues in this way *A* must win. For one of the piles must be reduced to zero by either *A* or *B*. If *B* reduces it to zero, the two remaining piles will be unequal, since *B* can never leave a safe combination, and *A* at his next move will make them equal, and will thereafter always leave them equal. *B* must, therefore, reduce the second pile to zero, and *A* then takes all of the third pile, and wins. If, on the other hand, *A* is the first player to reduce one of the piles to zero, he leaves the other two piles equal and wins as before. Hence we see that the player who can first leave a safe combination on the table should win.

If it happens that in the beginning a safe combination is placed on the table, the second player should win. If in the beginning a safe combination is not placed on the table, it is easily seen that the first player can always leave a safe combination by diminishing some one of the piles, and he can often do this by drawing from either one of the three piles. Therefore in this case the first player should win. That is, the first player should win or lose according as a safe combination is not or is placed on the table at the beginning.

3. The Chance of a Safe Combination. Assuming that the number in each pile at the beginning was determined by chance, let us compute the chance of a safe combination's being placed upon the table. It is easily shown that if each pile contains less than 2^n counters and if no pile is zero (*i. e.* if there are three piles), the possible number of different piles is

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[*] The proof of this statement depends on the fact that the number 100 . . . (n ciphers), or 2^n, is greater than the number 11 . . . (n ones), or $2^{n-1} + 2^{n-2} + \ldots + 2 + 1 = 2^n - 1$.

$$\frac{2^{n-1}(2^{2n}-1)}{3}.$$

The number of safe combinations in the same case is

$$\frac{(2^{n-1}-1)(2^{n}-1)}{3}.$$

Hence the chance of a safe combination's being placed upon the table at first is

$$\frac{2^{n-1}-1}{2^{n-1}(2^{n}+1)},$$

and this is the chance that the second player should win. The chances of the first player's winning are to those of the second as

$$2^{n}+2+\frac{3}{2^{n-1}-1} \text{ to } 1,$$

on the assumption that both players know the theory, and that the numbers in the various piles were determined by chance.

4. A List of Safe Combinations, $n = 4$. The following are the 35 safe combinations all of whose piles are less than 16:

1 2 3	2 4 6	3 4 7	4 8 12
1 4 5	2 5 7	3 5 6	4 9 13
1 6 7	2 8 10	3 8 11	4 10 14
1 8 9	2 9 11	3 9 10	4 11 15
1 10 11	2 12 14	3 12 15	
1 12 13	2 13 15	3 13 14	
1 14 15			

5 8 13	6 8 14	7 8 15
5 9 12	6 9 15	7 9 14
5 10 15	6 10 12	7 10 13
5 11 14	6 11 13	7 11 12

Of course, to give *all* safe combinations of numbers less than 16 we should have to add to the above table the 15 of the form $0, n, n$.

5. Generalization. The foregoing game can be at once generalized to the case of any number of piles, with the same rule for playing. In this case a safe combination is a set of numbers such that, when written in the binary scale and arranged with the units in the same vertical column, the sum of each column is even (*i. e.*, $\equiv 0$, mod. 2). Just as before, it is shown that the player who first leaves a safe combination can do so at every subsequent play, and will win. The induction proof is so direct that it seems unnecessary to give it.

6. Modification. The game may be modified by agreeing that the player who takes the last counter from the table *loses*. This modification of the three pile game seems to be more widely known than that first described, but its theory is not quite so simple.

A safe combination is defined just as in the first case, *except* that 1, 1, 0 is *not* a safe combination, and 1, 1, 1 and 1, 0, 0 *are* safe combinations. When the first theory indicates that A should play 1, 1, 0 he must play either 1, 1, 1 or 1, 0, 0. The earlier part of the proof proceeds as before. In order to complete it, we must show that B can never leave 1, 1, 1; that, when 1, 1, 0 is indicated for A, he can always play either 1, 0, 0 or 1, 1, 1; and finally that, if the play is carried out in this way, B must take the last counter. That B can never leave 1, 1, 1 is at once clear, for A never leaves 1, 1, n where $n > 1$, since this is not a safe combination. Secondly, let us consider what sets of numbers B can leave which would indicate 1, 1, 0 as A's next play in the first form of game. They are 1, 1, n where $n > 1$, and 1, n, 0 where $n > 1$. In the first case A leaves 1, 1, 1 and in the second 1, 0, 0. The proof is now easily completed. Either A or B reduces a pile to zero. If B does so, the other two piles are unequal and both greater than unity, or at least one of the two remaining piles is unity. In the latter case A obviously wins. In the former case A makes the two piles equal, and then keeps them equal until B reduces one of them to 1 or 0. If B makes it 1, A takes all the other pile; if B makes it 0, A takes all but 1 of the other pile. Hence if B first reduces a pile to zero A wins. If A first reduces a pile to zero he leaves the other two piles equal and each greater than unity, and wins as before. Hence if A plays on the safe combinations as here modified, B must take the last counter from the table, and *loses*. That is, in this modified game, also, the player who can first get a safe combination should win.

This modified game can also be generalized to any number of piles. The safe combinations are the same as before, *except* that an odd number of piles, each containing one, is now safe, while an even number of ones is not safe.

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ON THE GROUPS GENERATED BY TWO OPERATORS OF ORDER THREE WHOSE PRODUCT IS ALSO OF ORDER THREE.

By G. A. Miller.

Two operators of order two always generate a dihedral rotation group whose order is twice the order of their product.[*] In this case, the group generated by two operators is completely defined by the orders of the operators and that of their product. Similarly, the group generated by two operators of order three, whose product is of order two, is completely defined by the orders of the two generating operators and that of their product. That is, the alternating group of order 12 is the only one which satisfies these conditions.[†] On the contrary, two operators (s_1, s_2) of order three whose product is also of this order may generate any one of an infinite system of groups of a finite order. The present paper is devoted to the determination of some of the properties of the groups of this system.[‡]

The following conjugate commutators

$$s_3 = s_1^2 s_2^2 s_1 s_2, \qquad s_4 = s_2^2 s_1^2 s_2^2 s_1 s_2^2 = s_1 s_2 s_1^2 s_2^2, \qquad s_5 = s_2 s_1^2 s_2^2 s_1$$

of such a group (G)[§] are commutative since $s_3 s_4 s_5 = s_4 s_3 s_5 = 1$. The last equation may readily be verified by means of the equations $(s_1 s_2)^3 = s_1 s_2 s_1 s_2 s_1 s_2 = 1$ or $s_1 s_2 s_1 = s_2^2 s_1^2 s_2^2$. Hence s_3, s_4, s_5 generate an abelian subgroup (H) of G which is either cyclic or may be generated by two independent operators. We shall soon see that H is the commutator subgroup of G. From the equations

$$s_1^2 s_3 s_1 = s_1 s_2^2 s_1 s_2 s_1 = s_4, \qquad s_1^2 s_4 s_1 = s_2 s_1^2 s_2^2 s_1 = s_5, \qquad s_1^2 s_5 s_1 = s_3,$$

it follows that H is invariant in G. Its order (h) is the order (g) of G divided by either 3 or 9, since the group generated by s_2 and H contains $s_1^2 s_2^2 s_1 = s_2^2 s_5$ and hence is also invariant in G. This proves that H is the commutator subgroup of G.* Moreover, the given equations show that s_1 and s_2 transform the operators s_3, s_4, s_5 in the same manner. Having determined some of the properties which all of the groups under consideration possess, we proceed to determine some of the special properties which depend upon the factors of h and especially to determine the necessary and sufficient conditions that such groups exist.

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[*] *Bull. Amer. Math. Soc.*, vol. 7, 1901, p. 424.

[†] *Cf.* Burnside, *Theory of Groups of Finite Order*, 1897, p. 291.

[‡] *Cf. ibid.*, p. 296.

[§] In what follows the symbol G will always represent a group of the system under consideration.

When H is cyclic and of order p^a, p being any prime number, the matter is quite simple. If $p \equiv 2 \bmod 3$, it follows from the above equations that G is abelian.† In this case $s_3 s_4 s_5 = s_3^3 = 1$ and therefore $h = 1$. The non-cyclic group of order 9 is clearly the only group which satisfies these conditions. It is the only abelian group in the infinite system of groups under consideration and will not be considered in what follows. If $p \equiv 1 \bmod 3$, s_2 and H generate one and only one group of this system for every value of $a > 0$. Its order is $3h$ and it contains $2h$ operators of order 3. The direct product of any one of these groups and an operator of order 3 is of order $9h$ and belongs to this system.

That there is only one group of order $9h$ for every value of a may be proved as follows: Such a G is isomorphic to the group of order 3 with respect to those of its operators which are commutative with each operator of H. That is, G contains a cyclic invariant subgroup of order $3h$ and hence an invariant subgroup of order 3. It also contains an invariant subgroup of order $3h$, generated by H and s_2, which does not include any invariant group of order 3. In other words, it is the direct product mentioned above. Hence *there are two and only two groups for every value of $a > 0$ whenever $p \equiv 1 \bmod 3$.* When $p = 3$, it follows from $s_3 s_4 s_5 = 1$ that H is either the identity or of order 3.‡ In this case there is therefore only one group; viz. the non-abelian group of order 27 which includes no operator of order 9.

From what precedes, it follows that we can construct at least one group of the required system whenever H is cyclic and all the prime factors of h are $\equiv 1 \bmod 3$, or when h is three times such a number. Moreover, it is not difficult to see that the equation $s_3 s_4 s_5 = 1$ cannot be satisfied under any other conditions, since H is the direct product of its cyclic subgroups, each of whose orders is a power of a single prime. It remains to consider the cases when

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* *Quarterly Jour. of Math.*, vol. 28, 1896, p. 266; *Cf.* Weber, *Algebra*, vol. 2, 1899, p. 131.

† The group of isomorphisms of the cyclic group of order p^a is of order $p^{a-1}(p-1)$.

‡ Burnside, *l. c.*, p. 76.

H is non-cyclic, but is generated by two of its operators. We shall again begin with the case when $h = p^a$.

When $p \equiv 2 \bmod 3$, s_3 and s_4 must be the independent generators of H, since the cube of a common operator is identity. Similarly, we observe that s_3 and s_4 cannot have more than 3 common operators whenever $p = 3$. When $p \equiv 1 \bmod 3$, we may construct a G by establishing a (p^a, p^δ) isomorphism between two G's of orders $3p^a$, $3p^\delta$ respectively in such a manner that the resulting group contains subgroups of order p which are not invariant. Hence, in this case, the two invariants of H may have any orders such that their product is equal to h; *i. e.* s_3 and s_4 may have a common subgroup of any order less than the order of s_3.

It has been observed that the two invariants of H are equal when $p \equiv 2 \bmod 3$. It may therefore be assumed that the order of s_3 and s_4 is p^m, where $m = \frac{a}{2}$, and that H is of type (m, m). H contains $p^{m-1}(p+1)$ cyclic subgroups of order p^m, which may be divided into $p + 1$ sets such that each one of those of one set has only identity in common with any one of another set. These subgroups are transformed by the holomorph of H according to a transitive substitution group of degree $p^{m-1}(p+1)$ and of order $p^{3m-2}(p^2-1)$. Since $p \equiv 2 \bmod 3$, the group of isomorphisms of H contains operators of order 3 which permute each of its cyclic subgroups of order p^m.

Such an operator (s_1) transforms the $p + 1$ sets mentioned above as units. It cannot transform any one of these sets into itself, since such a set is composed of p^{m-1} groups of order p^m. Hence H and s_1 generate a group which contains $2h$ operators of order 3. For s_2 we may select any substitution which transforms the operators of H in the same manner as s_1 does and which gives a product of order p^m when multiplied by s_1. This proves the existence of the groups in question for every value of $p \equiv 2 \bmod 3$ and for every value of $m > 0$.

When $p = 3$ and the two independent generators (t_1, t_2) of H are either of order 3^m or of orders 3^m and 3^{m-1} respectively, it is easy to verify that there is an operator of order 3 in the group of isomorphisms of H, which transforms t_1 into $t_2 t_1$ and t_2 into $t_1^{-3} t_2^{-2}$. This operator, s_1, and H generate a group of order $3h$ which contains $2h$ operators of order 3 that are not found in H; for, $(t_1^{a} t_2^{\beta} s_1)^3 = t_1^{a} t_2^{\beta} t_2^{-a} t_1^{-3a} t_2^{\beta} t_1^{3\beta} t_2^{a} t_1^{a} t_1^{-3\beta} t_2^{-2\beta} = 1$. If we take for s_2 an operator such that $s_1^2 s_2 = t_1$, it is evident that s_1, s_2 generate G. This proves the existence of groups whenever $p = 3$ and H has two invariants of order 3^m or one invariant of order 3^m and the other of order 3^{m-1}. It was proved above that these conditions are also necessary. It remains, therefore, only to examine the case when H has two independent generators whose orders are not powers of the same prime number.

In this general case, we may write h in the form $3^{a_0}p_1^{a_1}p_2^{a_2}$ . . . and observe that each of the subgroups whose orders are $3^{a_0}, p_1^{a_1}, p_2^{a_2}$, . . . is either cyclic or the direct product of two cyclic groups.* The necessary and sufficient conditions that s_1 and s_2 can be so chosen as to generate a group which contains any one of these subgroups, as H, have been determined. If these conditions are satisfied for each of the given subgroups, we may suppose s_1 and s_2 so formed as to generate the group formed by establishing a $(3^{a_0}, p_1^{a_1}, p_2^{a_2}$ . . .) isomorphism between the separate group whose h's are powers of primes. If they are not satisfied in each instance, it follows from the equation $s_3 s_4 s_5 = 1$ that there is no group in the infinite system under consideration which has this H. Hence this general case is included under the special cases considered above.

CORNELL UNIVERSITY, JULY, 1901.

ON THE INVARIANTS OF A QUADRANGLE UNDER THE LARGEST SUBGROUP, HAVING A FIXED POINT, OF THE GENERAL PROJECTIVE GROUP IN THE PLANE.**

BY W. A. GRANVILLE.

IN the ANNALS OF MATHEMATICS, vol. 12, p. 82, Professor Lovett proposes the problem of finding the invariants of a quadrangle under the transformations of the six parameter group in the plane generated by the infinitesimal transformations :

$$\boxed{xp \quad yp \quad xq \quad yq \quad x^2p + xyq \quad xyp + y^2q},$$

$$p = \frac{\partial f}{dx}, \qquad q = \frac{\partial f}{dy}.$$

This is the largest subgroup of the general projective group in the plane, which has a fixed point. The invariants found by Professor Lovett were

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* *Cf. Bull. Amer. Math. Soc.*, vol. 7, 1901, p. 424.

** Presented to the American Mathematical Society at its meeting, April 27, 1901.

$$c_1 = \frac{y_3}{y_4} \cdot \frac{x_4 y_2 - x_2 y_4}{x_3 y_2 - x_2 y_3}, \qquad c_2 = \frac{x_2}{x_3} \cdot \frac{x_1 y_3 - x_3 y_1}{x_1 y_2 - x_2 y_1}.$$

On trial it will be found that these functions do not satisfy the differential equations of the complete system (6) on page 82:

$$\begin{aligned} &\sum_{i=1}^{4} x_i \frac{\partial\phi}{\partial x_i} = \sum_{i=1}^{4} y_i \frac{\partial\phi}{\partial x_i} = \sum_{i=1}^{4} x_i \frac{\partial\phi}{\partial y_i} = \sum_{i=1}^{4} y_i \frac{\partial\phi}{\partial y_i} = 0, \\ &\sum_{i=1}^{4} \left\{ x_i^2 \frac{\partial\phi}{\partial x_i} + x_i y_i \frac{\partial\phi}{\partial y_i} \right\} = \sum_{i=1}^{4} \left\{ x_i y_i \frac{\partial\phi}{\partial x_i} + y_i^2 \frac{\partial\phi}{\partial y_i} \right\} = 0; \end{aligned} \tag{6}$$

and hence cannot be the invariant functions.

The two solutions of (6) are found to be

$$c_1 = \frac{\left(\frac{y_1}{x_1} - \frac{y_3}{x_3}\right)\left(\frac{y_2}{x_2} - \frac{y_4}{x_4}\right)}{\left(\frac{y_1}{x_1} - \frac{y_4}{x_4}\right)\left(\frac{y_2}{x_2} - \frac{y_3}{x_3}\right)} = \frac{m_{01} - m_{03}}{m_{01} - m_{04}} \cdot \frac{m_{02} - m_{04}}{m_{02} - m_{03}};$$

and

$$c_2 = \frac{\left(\frac{y_1}{x_1} - \frac{y_1 - y_3}{x_1 - x_3}\right)\left(\frac{y_1 - y_4}{x_1 - x_4} - \frac{y_1 - y_2}{x_1 - x_2}\right)}{\left(\frac{y_1}{x_1} - \frac{y_1 - y_4}{x_1 - x_4}\right)\left(\frac{y_1 - y_3}{x_1 - x_3} - \frac{y_1 - y_2}{x_1 - x_2}\right)} = \frac{m_{10} - m_{13}}{m_{10} - m_{14}} \cdot \frac{m_{14} - m_{12}}{m_{13} - m_{12}};$$

where m_{ik} is the slope of the line drawn from the point (x_i, y_i) to the point (x_k, y_k), the origin being denoted by (x_0, y_0). Hence the theorem:

If a quadrangle (1234) *be transformed by the Lie group*

$$\boxed{xp \quad yp \quad xq \quad yq \quad x^2p + xyq \quad xyp + y^2q},$$

the cross ratio of the pencil of lines drawn from any vertex of the pentagon (01234) *to the remaining four vertices remains constant.*

If the problem be considered from the standpoint of projective geometry, this result follows at once from the well known fact that the cross ratio of a pencil of any four concurrent lines is an invariant under any projective transformation. It is also evident that two of the cross ratios here considered are independent.

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SOME APPLICATIONS OF THE METHOD OF ABRIDGED NOTATION.

By Maxime Bôcher.

1. Introduction. In proving a geometrical proposition by the method of abridged notation certain identities, often of an extremely simple character, are used, from whose geometric interpretation the proof of the theorem follows. It frequently happens that the set of identities we have used in the proof of our theorem admits of a second geometrical interpretation, and when we have noticed this fact, we have before us a new geometrical theorem which we have, so to speak, deduced from the theorem from which we started, or rather from our method of proving it.

To illustrate what I mean by a very simple example, let us consider the theorem that the bisectors of the angles of a triangle meet in a point. To prove this by the method of abridged notation we indicate the equations of the sides of the triangle by

$$(1) \qquad u = 0, \qquad v = 0, \qquad w = 0.$$

If we suppose that all these equations are written in the normal form:

$$x \cos \alpha + y \sin \alpha - p = 0,$$

the equations of the bisectors of the angles will be

$$(2) \qquad u - v = 0, \qquad v - w = 0, \qquad w - u = 0;$$

and now the identity:

$$(3) \qquad (u - v) + (v - w) + (w - u) \equiv 0$$

shows us at once that these lines meet in a point.

Suppose, however, instead of interpreting the equations (1) as the equations of three straight lines, we regard them as the equations of three circles all written in the normal form:

$$x^2 + y^2 + ax + by + c = 0.$$

Then the equations (2) will represent the common chords of these circles, and the identity (3) tells us that these common chords meet in a point.

Thus we see that the method of abridged notation may be used to deduce new theorems from known ones, just as the methods of projection, reciprocation, and inversion are commonly used. As this fact has, in spite of its elementary nature, gained as yet no place, so far as I have found, in even the best text books,[*] I propose to illustrate it here by first reproducing the well known proofs by the method of abridged notation of the harmonic properties of the complete quadrilateral and of Desargues's theorem, and then interpreting these proofs in such a way as to obtain theorems which, instead of referring to rectilinear figures, as did the original propositions, refer to much more general curvilinear figures. In doing this I have paid special attention to the case in which these curved lines are circles, not only because this is the simplest case, but because the theorems thus obtained are suited to hold a fundamental position, as I expect to show on another occasion, in a systematic development of what has been termed the Geometry of Inversion.[†] It must, however, be clearly understood that the generalizing process which we use in this simple case is not identical with the process of inversion, but that the results obtained include as a very special case the theorems which the method of inversion would give us.

2. Rectilinear Figures. Let us denote the sides of a complete quadrilateral by the equations:

$$\alpha = 0, \qquad \beta = 0, \qquad \gamma = 0, \qquad \delta = 0. \tag{4}$$

Denoting the six vertices by

$$(\alpha, \beta), \quad (\gamma, \delta), \quad (\alpha, \gamma), \quad (\beta, \delta), \quad (\alpha, \delta), \quad (\beta, \gamma), \tag{5}$$

we will let

$$\begin{array}{ll} u = 0 & \text{represent the diagonal joining } (\beta, \gamma) \text{ and } (\alpha, \delta), \\ v = 0 & \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad (\alpha, \gamma) \text{ "} (\beta, \delta), \\ w = 0 & \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad (\alpha, \beta) \text{ "} (\gamma, \delta). \end{array} \tag{6}$$

---

[*] Except in the case where we merely pass from point to line coordinates,—a case which is essentially equivalent to reciprocation.

[†] *Cf.* Klein's *Erlanger Programm* (1872), reprinted in English translation and with a few additions in the *Bulletin of the New York Mathematical Society* for July 1893. A German reprint with still further additions will be found in the *Math. Annalen*, vol 43 (1893), p. 63.

The values of u, v, w, in terms of α, β, γ, δ may be obtained as follows:— We may write the equation of the diagonal u in either of the forms

$$b\beta + c\gamma = 0, \qquad -a\alpha - d\delta = 0.$$

If we so choose the constants a, b, c, d that the first members of these equations are equal, not merely proportional, we get the identity:

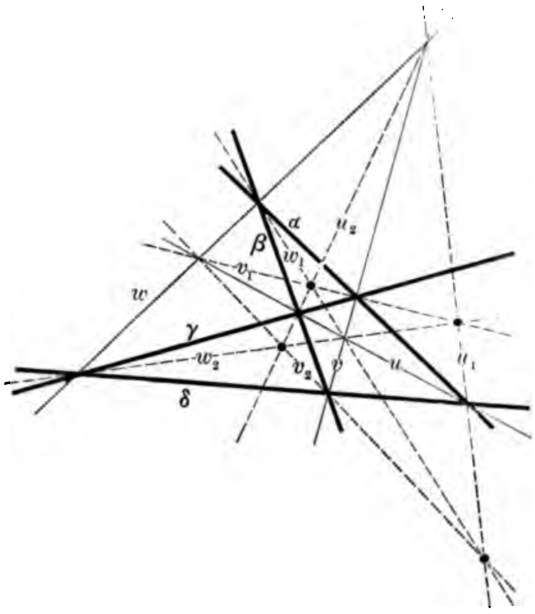

FIG. 1.

$$(7_1) \qquad u \equiv b\beta + c\gamma \equiv -a\alpha - d\delta.$$

From this it follows that

$$b\beta + d\delta \equiv -a\alpha - c\gamma.$$

If we set this expression equal to zero we get, as we see from the left hand side, a line through (β, δ), as we see from the right hand side, a line through (α, γ). We may therefore write:

$$(7_2) \qquad v \equiv b\beta + d\delta \equiv -a\alpha - c\gamma.$$

Similarly we get:

$$(7_3) \qquad w \equiv c\gamma + d\delta \equiv -a\alpha - b\beta.$$

Finally indicating the lines which join the vertices of the quadrilateral with the points of intersection of the diagonals, as shown in the figure, by:

$$u_1 = 0, \qquad u_2 = 0, \qquad v_1 = 0, \qquad v_2 = 0, \qquad w_1 = 0, \qquad w_2 = 0, \tag{8}$$

we have

$$\begin{aligned} u_1 &\equiv v + w \equiv -a\alpha + d\delta, \\ u_2 &\equiv v - w \equiv b\beta - c\gamma, \\ v_1 &\equiv w + u \equiv -a\alpha + c\gamma, \\ v_2 &\equiv w - u \equiv d\delta - b\beta, \\ w_1 &\equiv u + v \equiv -a\alpha + b\beta, \\ w_2 &\equiv u - v \equiv c\gamma - d\delta. \end{aligned} \tag{9}$$

From these formulæ we read off at once the harmonic properties of the complete quadrilateral, for instance that the two sides α and β are separated harmonically by the diagonal w through their point of intersection and the line w_1 which joins their point of intersection with the intersection of the two other diagonals (*cf.* (7_3) and the fifth formula (9)).

We may add that formulæ (9) also show that the six lines (8) pass three by three through four points, thus forming the sides of a complete quadrangle.

We pass now to Desargues's theorem:—

If two triangles are so situated that the points of intersection of corresponding sides are collinear, then the lines joining corresponding vertices are concurrent.

To prove this theorem we denote the sides of the first triangle by:

$$u_1 = 0, \qquad u_2 = 0, \qquad u_3 = 0; \tag{10}$$

the sides of the second by:

$$v_1 = 0, \qquad v_2 = 0, \qquad v_3 = 0. \tag{11}$$

The line on which the points of intersection of corresponding sides lie may be represented by an equation in any one of the following forms:

$$a_1u_1 + b_1v_1 = 0, \qquad a_2u_2 + b_2v_2 = 0, \qquad a_3u_3 + b_3v_3 = 0.$$

Accordingly, by properly choosing the constants, we can make the first members of these equations identically equal:

$$a_1u_1 + b_1v_1 \equiv a_2u_2 + b_2v_2 \equiv a_3u_3 + b_3v_3. \tag{12}$$

From this we get at once the following three identities:

$$
\begin{aligned}
w_3 &\equiv a_1u_1 - a_2u_2 \equiv -b_1v_1 + b_2v_2,\\
w_1 &\equiv a_2u_2 - a_3u_3 \equiv -b_2v_2 + b_3v_3,\\
w_2 &\equiv a_3u_3 - a_1u_1 \equiv -b_3v_3 + b_1v_1.
\end{aligned} \tag{13}
$$

From these identities we infer that the equations:

$$w_1 = 0, \qquad w_2 = 0, \qquad w_3 = 0 \tag{14}$$

represent the lines joining corresponding vertices of the two triangles, and since

$$w_1 + w_2 + w_3 \equiv 0, \tag{15}$$

these lines are concurrent.

3. Generalization to Circles. Let us now try to interpret the analytical work of the preceding section on the supposition that the equations there used represent not straight lines, but circles.

We begin with the theorems about the complete quadrilateral.

The figure formed by four circles, no three of which meet in the same two points, we will call a complete circular quadrilateral, the circles (4) being the *sides* of the quadrilateral and the six pairs of points (5) being its six *sets of vertices*. These latter arrange themselves in pairs of *opposite* sets of vertices. The equations $b\beta + c\gamma = 0$ and $-a\alpha - d\delta = 0$ now represent circles through the pairs of points (β, γ) and (α, δ) respectively, but it will not be possible in general to make these two circles coincide, since a circle cannot in general be passed through two pairs of points.

In order that our analytic work be applicable to this case it is therefore necessary that the four points (β, γ) and (α, δ) be concyclic (*i. e.* lie on a circle). This circle will then be represented by the equation $u = 0$ (*cf.* (7_1)). From this equation we infer as before that $v = 0$ and $w = 0$ (*cf.* (7_2) and (7_3)) represent circles through the two pairs of points (α, γ), (β, δ) and (α, β), (γ, δ) respectively. We thus get the theorem:

*If in a complete circular quadrilateral two opposite sets of vertices are concyclic, the same will be true of each of the other pairs of opposite sets of vertices.**

If a circle passes through a pair of opposite sets of vertices of a complete

---

* Another form of statement for this theorem which brings it into relation with Desargues's theorem is the following, in which we denote the figure formed by two circles as a *crescent*, the two circles being the *sides* and their points of intersection the *vertices*:

If two crescents are so situated that their vertices are concyclic, the points of intersection of circular quadrilateral, we will call it a diagonal of this quadrilateral. We may then state the last theorem as follows:

If a complete circular quadrilateral has one diagonal, it has three.

We will recall here two terms which we shall find it convenient to use. $S_1 = 0$ and $S_2 = 0$ being any two given circles, we call the simply infinite family

$$(16) \qquad k_1 S_1 + k_2 S_2 = 0$$

a *pencil* of circles; and we use the same term for a finite number of circles of this family. Similarly $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ being any three circles which do not belong to a pencil the doubly infinite family:

$$(17) \qquad k_1 S_1 + k_2 S_2 + k_3 S_3 = 0$$

we call a *net.*[*]

We now infer at once from formulæ (7):

A necessary and sufficient condition that a complete circular quadrilateral have diagonals is that its sides belong to a net. If this condition is fulfilled, the diagonals belong to the same net.

Under these conditions any two sides of the quadrilateral (as α, β) and the two diagonals which do not pass through the intersections of these sides (as u, v) form a complete circular quadrilateral whose sides belong to a net and which therefore has diagonals. Two of these diagonals are the other two sides (γ, δ) of the original quadrilateral, but the third is the circle (w_1) through the points of intersection of α with β and of u with v. Similarly we obtain the other circles (8).

By the *cross ratio* of a pencil of four circles we shall understand the cross ratio of the pencil of tangents at any one of their points of intersection.[†] With

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corresponding sides (the correspondence being taken in either one of the two possible ways) are also concyclic.

Still another form, in which, however, only half the theorem is explicitly stated, is the following:

Through each of the sets of vertices of a circular triangle a circle is passed. If these three circles meet in a first point, they will meet again in a second point.

As thus stated the theorem is seen to be identical with the generalization by inversion of the theorem that the common chords of three circles meet in a point.

[*] We mention in passing that such a net consists of all circles which cut at right angles the common orthogonal circle of S_1, S_2, S_3. *Cf.* for instance Casey's *Analytical Geometry*, p. 107.

[†] Or more generally the cross ratio of the pencil formed by the polars of an arbitrarily chosen point. This more general definition is necessary if the circles of the pencil touch each other.

this definition it is readily seen that a necessary and sufficient condition that the circles S_1 and S_2 be divided harmonically by $S_1 + \lambda S_2$ and $S_1 + \mu S_2$ is that $\lambda = -\mu$. We thus obtain, precisely as in §1, the harmonic properties of the complete circular quadrilateral, of which the following is typical:

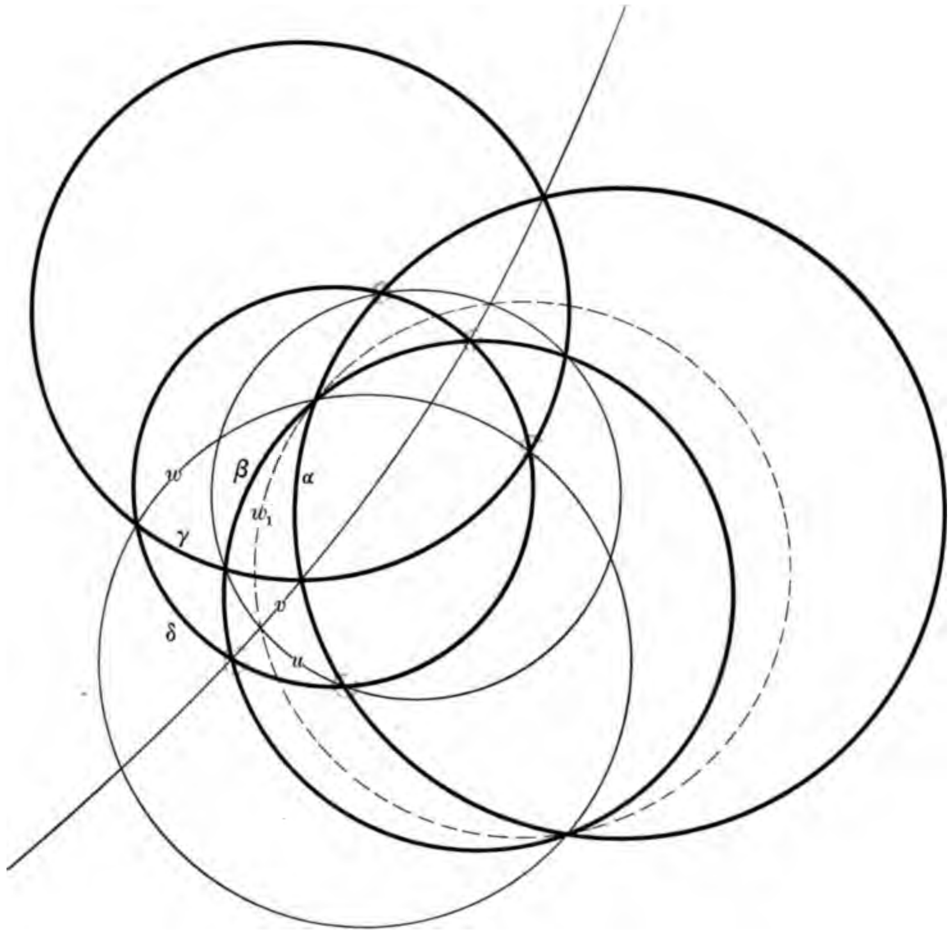

FIG. 2.

In a complete circular quadrilateral whose sides belong to a net, any two sides are divided harmonically by the diagonal through their points of intersection, and the circle through their points of intersection and through the points of intersection of the other two diagonals.

We leave to the reader the extension of the theorem that the lines (8) form a complete quadrangle.

Up to this point all the circles we have used have belonged to a single net;* and it might seem at first sight as though the method we are using must necessarily always give us theorems which involve only circles of a single net, since the straight lines from which these circles were obtained themselves form a net. This, however, is not the case, as the extension of Desargues's theorem will now show. For although any four lines of the plane are connected by a relation of the form $a\alpha + b\beta + c\gamma + d\delta \equiv 0$, it is not at all necessary that this relation be involved in the formulæ we use in proving our proposition, as is shown by the proof we have given of Desargues's theorem.†

Turning now to the proof of Desargues's theorem given in §1 we have to consider two circular triangles (10) and (11). We understand of course by a circular triangle the figure formed by three circles which do not form a pencil. Such a triangle has three sides and three pairs of vertices. The analysis of §1 then gives us the theorem:

If two circular triangles are so situated that the three pairs of points of intersection of corresponding sides are concyclic, then any pair of vertices of the first triangle and the corresponding pair of vertices of the second are concyclic,‡ and the three circles thus determined form a pencil.

The ten circles involved in this theorem do not necessarily all belong to a net, since the three sides of the first triangle and one side of the second may be arbitrarily chosen.

4. Curves of the n-th Order. We will now regard the equations from which we started as representing plane curves of the nth order instead of straight lines as in §2 or circles as in §3. We will speak of pencils and nets of curves of the nth order, using the words in the ordinary sense similar to that explained for circles in §3. We will speak of four curves of the nth order, no three of which form a pencil, as the *sides* of a *complete curvilinear quadrilateral.*[*] Each pair of sides of this quadrilateral determines a pencil, and if the pencils determined by two opposite pairs of sides have a curve in common, we will speak of this curve as a *diagonal* of the quadrilateral. Formulæ (7) now establish the following theorem:

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* In fact our formulæ practically establish a one-to-one transformation of the straight lines of the plane into the circles of a net, and therefore a one-to-two point transformation (a point determined by two straight lines into the pair of points determined by the corresponding circles). Such a correspondence can easily be established geometrically by projecting the plane from an arbitrarily chosen point onto an arbitrarily chosen spherical surface, and then projecting back stereographically onto the plane.

† A different proof of this theorem which does not have this advantage is given in Salmon's *Conic Sections*, p. 59.

‡ This first part of the theorem does not go beyond what we have already proved, since two pairs of corresponding sides of the triangle form a complete circular quadrilateral lying wholly in a net.

A necessary and sufficient condition that a complete curvilinear quadrilateral have a diagonal is that its sides belong to a net, and if this condition is fulfilled, it has three and only three diagonals which also belong to the net.

Let us now consider the pencil of four curves

$$S_1 = 0, \qquad S_2 = 0, \qquad S_1 + \lambda S_2 = 0, \qquad S_1 + \mu S_2 = 0.$$

The linear polars of any point P of the plane, excluding possibly certain exceptional positions for P, are easily shown to form a pencil, and the cross ratio of this pencil is seen to be independent of the position of P and to have the value λ/μ. This quantity we will speak of as the cross ratio of the pencil of curves.[†] In particular we may in general[‡] take for P a point of intersection of the curves, in which case the cross ratio of the pencil of curves is given by the cross ratio of the pencil of tangents at this point of intersection.

If $\lambda = -\mu$ we say that the curves S_1 and S_2 are *divided harmonically* by the other two curves.

Formulæ (7) and (9) now give us a number of theorems concerning the complete curvilinear quadrilateral, of which we state the following one:

If a complete curvilinear quadrilateral is formed by four curves of a net, any two sides α, β of the quadrilateral are divided harmonically by the diagonal belonging to the pencil (α, β) and the curve common to the pencil (α, β) and the pencil determined by the other two diagonals.

The extension to this case of the further theorems contained in §2 we leave to the reader.

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[*] Such a quadrilateral has in general six sets of n^2 vertices each. In case some of the sides are reducible curves two sides may, however, have a whole curve in common. For this reason we have avoided speaking of vertices in the text.

[†] If instead of considering the linear polars of P we consider the polars of any order, it is clear that these will also form a pencil having the same cross ratio as that of the original pencil.

[‡] Provided, namely, there exists a point of intersection where neither S_1 nor S_2 has a double point, and where these curves do not touch each other.

5. Extension to Surfaces. It is hardly necessary to do more than mention the possibility of interpreting the analysis of §2 on the supposition that the equations there used represent surfaces in space instead of lines or curves in the plane. Thus, for instance, if we regard the equations as representing spheres, the formulæ concerning the complete quadrilateral in §2 give us certain theorems concerning spheres which have two points in common, while the proof of Desargues's theorem gives us a theorem concerning ten spheres which cut a given sphere orthogonally. The statement of these theorems and of similar theorems concerning surfaces of any order presents no difficulty.

Even without changing the degrees of the equations we use, the method here considered gives us the means of transferring theorems in plane geometry to the case of three dimensions. The theorems of §2 are too simple to give anything interesting in this way; but it will be a good exercise for a student to transfer by this method to quadric surfaces the theorem that *if three conics have double contact with a fourth, six of their chords of intersection pass three by three through four points*, using for this purpose the proof by abridged notation given in Salmon's *Conic Sections*, p. 243.

It will be seen that examples of cases to which our method applies can be multiplied indefinitely. We have confined ourselves here to a few of the very simplest illustrations.

HARVARD UNIVERSITY,
NOVEMBER 1901.

ON THE ROOTS OF FUNCTIONS CONNECTED BY A LINEAR RECURRENT RELATION OF THE SECOND ORDER.*

BY M. B. PORTER.

AT the close of his first memoir on homogeneous linear differential equations of the second order in the first volume of Liouville's Journal, Sturm remarks that results analogous to those obtained for the differential equation were first discovered by him while considering solutions of a homogeneous equation in finite differences of the form,

$$(1) \qquad L_n y_{n+1} + M_n y_n + N_n y_{n-1} = 0,$$

when L, M, and N are functions, subject to certain restrictions, of the positive integral index n and a variable parameter, and that it was while prosecuting this inquiry that he discovered his famous theorem concerning the isolation of the roots of a polynomial. By passing from the discrete index n to the continuous variable x, Sturm first arrived at the theorems for the differential equation in the paper referred to. This earlier method of dealing with the problems solved by Sturm was perhaps regarded by him as merely heuristic, or at least as less elegant than the methods finally employed, and his earlier researches were never published.

Characterizing a sequence of real functions $y_0, y_1, \ldots y_n$ as *Sturmian*† when the difference in the number of variations of sign in the sequence for two particular real values of the argument is equal to the number of real roots of y_n in the interval delimited by them, it is certain that Sturm determined at least one simple sufficient condition to which a sequence must conform in order that it may be *Sturmian*, and that he investigated the inverse problem of constructing such sequences by means of a recurrent relation of type (1). Looked at from the standpoint of the recurrent relation, solutions of (1) enjoying the *Sturmian* property may be called *Sturmian solutions*; undoubtedly Sturm discovered many properties of such solutions.

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* Read before the American Mathematical Society, February, 1901, under a different title.

† This adjective is here used in a more general sense than in the ensuing sections. *Cf.* the definitions there given.

It is our purpose in the first five sections to reproduce in part these unpublished theorems and in the last section to show how, by means of the Cauchy-Lipschitz theorem for the existence of solutions of a differential equation, it is possible to establish rigorously the analogous theorems, so far as they exist, for the homogeneous linear differential equation of the second order. As has already been stated, in this case the index n is replaced by the independent variable and the variable parameter either figures in the differential equation itself, or in the arbitrary parameters of integration, or in both.

In conclusion it may be said that it is highly probable that Sturm considered a somewhat more general difference equation than (1′) of §3 and so arrived at criteria of a more general character. This point, however, I hope to consider elsewhere.

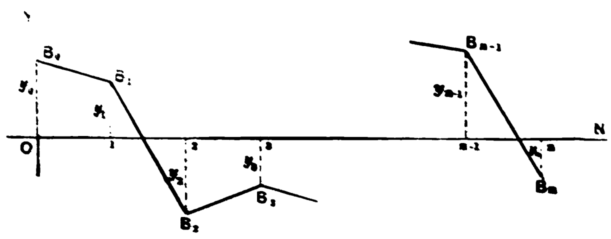
1. Sturmian Sequences in General. We begin by considering a series of functions

$$y_n(x), \quad y_{n-1}(x), \; . \; . \; . \; . \; . \; y_0(x)$$

of the real variable x, real, single valued, and analytic in an interval $x_0 \leqq x \leqq x_1$.*

A. *It is further supposed that for no point of this interval is $y_0(x)$ equal to zero.*

It will be convenient to employ the following graph in considering the



above sequence. The ordinate $i\,B_i$ representing y_i in magnitude and sign, the abscissa Oi may conveniently be taken equal to the index i. In the graph just described, the intersections of the broken line $B_0\,B_1 \; . \; . \; . \; . \; B_n$ with the N-axis, which we shall call *v-points*, correspond to the *variations of sign* in the sequence $y_0, y_1, \; . \; . \; . \; y_n$.

If now we suppose the parameter x to vary continuously, this broken line will undergo a continuous deformation, the point B_0 remaining always above the origin, while the v-points are displaced in a manner which will depend on the particular functions $y_1, \ldots y_n$. It is evident that at least one variation will be lost or gained when y_n changes sign. The vanishing of an intermediate y can cause the loss or gain of a variation only when $B_0 B_1 \ldots . B_n$ is deformed in such a manner that, when a given y vanishes, the adjacent y's have the same sign, or one of the adjacent y's is zero. In order that the number of variations of sign in the sequence $y_0, y_1, \ldots . y_n$ should change as x varies from x_0 to x_1 *only* when y_n changes sign, it will be *sufficient*, if we impose the condition :

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* Throughout the rest of this paper we shall suppose that the variable x is confined to such an interval.

B. No two consecutive y's can vanish for the same value of x, and when any intermediate y vanishes the adjacent y's have opposite signs.

A series of functions of the kind specified, *i. e.* fulfilling Conditions A and B, are said to form a *generalized Sturmian* sequence.*

Let us now consider what farther restrictions must be imposed on our sequence in order that it should lose a variation each time y_n vanishes. Let x' be a value of x for which $y_n = 0$. The sequence will lose a variation as x increases through the value x' when and only when y_n and y_{n-1} have opposite signs for values of x a little smaller than x' and the same sign for values a little larger, *i. e.* when

$$(2) \qquad \frac{y_n(x'-\Delta x)}{y_{n-1}(x'-\Delta x)} < 0 < \frac{y_n(x'+\Delta x)}{y_{n-1}(x'+\Delta x)} \qquad \left(\begin{array}{l}\Delta x = \text{a sufficiently small} \\ \text{positive quantity.}\end{array}\right)$$

From (2) we see that y_n/y_{n-1} is increasing with x at the point x', and therefore we have as a consequence of (2) :

$$(3) \qquad \left[\frac{d}{dx}\left(\frac{y_n}{y_{n-1}}\right)\right]_{x=x'} \geqq 0.$$

On the other hand, if a variation is to be gained :

$$\left[\frac{d}{dx}\left(\frac{y_n}{y_{n-1}}\right)\right]_{x=x'} \leqq 0.$$

We shall treat explicitly only the case when variations are lost as x increases.

It is to be noted that condition (2) is more general than (3) and that we cannot pass back to (2) except when the inequality sign holds. It is thus a sufficient, but not a necessary, condition for the loss of a variation that*

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* *Cf.* Netto, *Algebra*, vol. 1, p. 238.

$$(4) \qquad \frac{d}{dx}\left(\frac{y_n}{y_{n-1}}\right) = \frac{y_n'}{y_{n-1}} > 0 \qquad \text{when } y_n = 0.$$

When this condition is fulfilled as well as conditions A and B, the sequence is called a *Sturmian* sequence.† It should be noted that condition (4) makes it impossible for y_n to have a multiple root in the interval (x_0, x_1) considered.

Starting with a real polynomial which has no multiple roots in (x_0, x_1) and choosing $y_{n-1} \equiv y_n'$, Sturm's modified algorithm of the Greatest Common Divisor leads to a sequence possessing these properties, the ordinary Sturm's Functions of text books on the Theory of Equations.

2. The Difference Equation. If y_n denote a function of the integral index n, where we shall always suppose n to denote a positive integer or zero, y_n is said to satisfy a *homogeneous linear recurrent relation of the second order* if we have for all values of n,

$$(1) \qquad L_n y_{n+1} + M_n y_n + N_n y_{n-1} = 0,$$

where L_n, M_n, and N_n denote functions of the index n. We will assume in what follows that none of the functions L_n is zero.

If we write

$$\begin{aligned} y_{n+1} - y_n &= \Delta y_n, \\ y_{n+2} - y_{n+1} &= \Delta y_{n+1}, \end{aligned}$$

and

$$\Delta y_{n+1} - \Delta y_n = \Delta^2 y_n,$$

relation (1) may be written in the form of a difference equation

$$L_n' \Delta^2 y_n + M_n' \Delta y_n + N_n' y_n = 0,$$

a form entirely analogous to that of the homogeneous linear differential equation of the second order. We shall however employ this equation only in the recurrent form.

If to y_0 and y_1 be assigned any values we choose,‡ all subsequent y's are uniquely determined by means of (1), since we have assumed that L_n does not vanish; so that (1) may be said to have ∞^2 solutions.

If $y_n^{(1)}$ and $y_n^{(2)}$ denote any two solutions, then, $C_1 y_n^{(1)} + C_2 y_n^{(2)}$ is also a solution.* $y_n^{(1)}$ and $y_n^{(2)}$ are said to be linearly dependent if a relation of the form

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* If multiple roots are excluded (4) is of course as general as (2).

† *Cf.* Weber, *Algebra*, vol. 1, p. 272 (1st ed.).

‡ This is analogous to the determination of a solution of a differential equation of the second order by its value and the value of its derivative at a given point.

$$C_1 y_i^{(1)} + C_2 y_i^{(2)} = 0, \qquad i = 0,\ 1,\ 2 \ldots n$$

exist between them, where the C's denote constants.

We will now establish the theorem:

The necessary and sufficient condition for the linear dependence of two solutions $y_n^{(1)}$ *and* $y^{(2)}$ *is*

$$y_0^{(1)} y_1^{(2)} - y_0^{(2)} y_1^{(1)} = 0. \tag{5}$$

For if $y_n^{(1)}$ and $y_n^{(2)}$ are linearly dependent, we have:

$$\begin{aligned} C_1 y_0^{(1)} + C_2 y_0^{(2)} &= 0, \\ C_1 y_1^{(1)} + C_2 y_1^{(2)} &= 0, \end{aligned} \tag{6}$$

where C_1 and C_2 are not both zero, and therefore (5) is fulfilled.

Conversely, if (5) is fulfilled, there exist two constants C_1 and C_2 not both zero which satisfy (6). Now if we have:

$$\begin{aligned} C_1 y_k^{(1)} \quad + C_2 y_k^{(2)} \quad &= 0, \\ C_1 y_{k-1}^{(1)} + C_2 y_{k-1}^{(2)} &= 0, \end{aligned}$$

then by equation (1) we have at once,

$$C_1 y_{k+1}^{(1)} + C_2 y_{k+1}^{(2)} = 0.$$

Therefore starting from (6) we see that:

$$C_1 y_n^{(1)} + C_2 y_n^{(2)} = 0 \qquad (n = 2,\ 3,\ 4,\ \ldots\ldots).$$

If then $y_n^{(1)}$ and $y_n^{(2)}$ denote any two linearly independent particular solutions and y_n the general solution, and if we determine C_1 and C_2 by means of the equations,

$$\begin{aligned} y_0 &= C_1 y_0^{(1)} + C_2 y_0^{(2)}, \\ y_1 &= C_1 y_1^{(1)} + C_2 y_1^{(2)}, \end{aligned}$$

then

$$y_n = C_1 y_n^{(1)} + C_2 y_n^{(2)} \qquad (n = 0,\ 1,\ 2,\ 3,\ \ldots\ldots)$$

i. e. any solution can be expressed linearly in terms of any two particular linearly independent solutions.

In the next section we shall be mainly concerned with a recurrent relation of the form,

$$y_{n+1} + G_n y_n + y_{n-1} = 0. \tag{1'}$$

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* Analogous to the two arbitrary constants in the general solution of a differential equation of the 2nd order.

Equation (1) may be reduced to this form by the change of dependent variable

$$y_n = e_n z_n. \tag{7}$$

Thus (1) becomes

$$L_n e_{n+1} z_{n+1} + M_n e_n z_n + N_n e_{n-1} z_{n-1} = 0.$$

If now we determine e_n by the recurrent relation

$$L_n e_{n+1} = N_n e_{n-1}$$

we get an equation of the desired form, where e_0 and e_1 may be chosen at pleasure, subject merely to the restriction that they do not vanish, and

$$e_{2n} = \frac{N_{2n-1} N_{2n-3} \cdot \cdot \cdot \cdot N_1}{L_{2n-1} L_{2n-3} \cdot \cdot \cdot \cdot L_1} e_0 . \qquad e_{2n+1} = \frac{N_{2n} N_{2n-2} \cdot \cdot \cdot \cdot N_2}{L_{2n} L_{2n-2} \cdot \cdot \cdot \cdot L_2} e_1,$$

$$G_n = \frac{M_n}{L_n} \frac{e_n}{e_{n+1}} .$$

3. Sturmian Sequences satisfying a Recurrent Relation of Type (1).

We shall now consider a sequence of functions y_n ($n = 0, 1, . . n$) satisfying a relation of the form,

$$L_n y_{n+1} + M_n y_n + N_n y_{n-1} = 0.$$

in which L, M, and N are functions of the index n and are real, single valued, and analytic functions of a real variable x in an interval $x_0 \leqq x \leqq x_1$. If, as we shall assume to be the case, $L_n(x)$ does not vanish in this interval, y_2 and all y's of higher index will be determined as real single valued analytic functions of x in this interval as soon as y_0 and y_1 are assigned as arbitrary real analytic functions of x.

Let us now choose y_0 so that it will not vanish in the interval (x_0, x_1) and seek to determine what conditions may be imposed on L, M, N, y_0, and y_1, so that the sequence $y_0, y_1, . . y_n$ shall form a Sturmian sequence. To ensure that no two consecutive y's shall vanish for the same value of x and that, when any y vanishes, the adjacent y's shall have opposite signs, it will be sufficient if we suppose that *not only L but also N does not vanish in the interval* (x_0, x_1) *and that these functions have like signs, i. e.* that LN is always positive. If these conditions are satisfied, the y's form a generalized Sturmian sequence.

To determine a sufficient condition that

$$\left[\frac{d}{dx}\left(\frac{y_n}{y_{n-1}}\right)\right]_{x=x'} > 0,$$

when x' denotes a root of y_n in the interval (x_0, x_1), we shall suppose (1) reduced to the form (1′) by the transformation (7) of §2 and consider the equation as given in the form,

$$(1') \qquad y_{n+1} + G_n y_n + y_{n-1} = 0,$$

when evidently the condition $LN > 0$ is fulfilled.* The relation between z_n and y_n (§2) shows that, under the restrictions placed on L and N, z and y vanish together.

Let x take the positive increment Δx, and let us write $y_i(x + \Delta x) = \bar{y}_i$, etc. Then (1′) becomes

$$\bar{y}_{n+1} + \bar{G}_n \bar{y}_n + \bar{y}_{n-1} = 0.$$

So that, when $y_n \bar{y}_n$ is not zero, we have, since $y_0(x) \neq 0$:

$$\Delta \frac{y_{n+1}}{y_n} = \frac{\bar{y}_{n+1}}{\bar{y}_n} - \frac{y_{n+1}}{y_n} = \frac{1}{\bar{y}_n y_n}\left[(-\bar{G}_n + G_n)\,\bar{y}_n y_n + \bar{y}_n y_{n-1} - y_n \bar{y}_{n-1}\right]$$

$$(8) \qquad = \frac{1}{\bar{y}_n y_n}\left[\sum_1^n (-\bar{G}_m + G_m)\,\bar{y}_m y_m + \bar{y}_0 y_0 \left(\frac{\bar{y}_1}{\bar{y}_0} - \frac{y_1}{y_0}\right)\right].$$

Since all the y's are continuous, we can, for a given value of x, take Δx so small that

$$\bar{y}_m y_m \geqq 0, \qquad m = 0, 1, 2, \ldots n.$$

Let us now impose the restrictions:

$$\left.\begin{array}{ll} 1^\circ & \bar{G}_m \leqq G_m \\ & {\scriptstyle m = 1, 2 \ldots n.} \\ 2^\circ & \bar{y}_1/\bar{y}_0 \geqq y_1/y_0 \end{array}\right\} \qquad x_0 \leqq x \leqq x_1$$

where, if the equality sign holds in 2°, we require either that there be two successive relations 1° where it does not hold, or that it does not hold in the last relation 1°.

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* It may be remarked in passing that this transformation is unnecessary if N/L is independent of x.

Under these circumstances a quantity δ can be found such that, if $y_n(x) \neq 0$,

$$\Delta \frac{y_{n+1}}{y_n} > 0 \qquad \text{when } \Delta x < \delta.$$

Thus y_{n+1}/y_n will then increase continually with x from $-\infty$ to $+\infty$ in every interval of (x_0, x_1) delimited by two consecutive roots of $y_n(x)$. Since the y's are analytic functions of x, the limits

$$\lim_{\Delta x = 0} \frac{\Delta\left(\frac{y_{n+1}}{y_n}\right)}{\Delta x}, \qquad \lim_{\Delta x = 0} \frac{\Delta\left(\frac{y_1}{y_0}\right)}{\Delta x}$$

exist and we have,

$$(9) \qquad \left(\frac{y_{n+1}}{y_n}\right)' = \frac{1}{y_n^2}\left[\sum_{1} - G'_m y_m^2 + y_0^2 \left(\frac{y_1}{y_0}\right)'\right] \qquad y_n \neq 0.$$

The conditions

$$1^\circ \quad G'_m(x) \leqq 0,$$

$$2^\circ \quad \left(\frac{y_1}{y_0}\right) \geqq 0, \qquad m = 1, 2, \ldots n$$

the restrictions above stated as to the simultaneous presence of equalities in 1° and 2° holding, yield at once the inequality

$$\left(\frac{y_{n+1}}{y_n}\right)' > C > 0, \qquad y_n \neq 0,$$

when x is supposed to lie in an interval included within an interval delimited by two consecutive roots of y_n. Thus, if conditions 1° and 2° are fulfilled, the functions $y_0, y_1, y_2 \ldots y_n$ form a Sturmian sequence. We may call such solutions *Sturmian solutions.*

Let us now consider the case in which G'_n and $y\ /y_0$ are real analytic functions of several parameters $x_1, x_2, \ldots$ in the intervals

$$x_0^{(i)} < x_i < x_1^{(i)}, \qquad i = 1, 2, \ldots.$$

We will assume that these functions satisfy the conditions

$$1^\circ \quad \frac{\partial}{\partial x_i} G'_m \leqq 0, \qquad m = 1, 2, \ldots n$$

$$2^\circ \quad \frac{\partial}{\partial x_i}\left(\frac{y_1}{y_0}\right) \geqq 0. \qquad i = 1, 2, \ldots.$$

Let us first consider the case in which there are only two parameters x_1, x_2, and ask ourselves how the roots of y_{n+1}, regarded as a function of x_1, vary with x_2. It is clear that these roots are continuous (in fact analytic) functions of x_2 which, since they are never multiple roots, can cease to be real only by disappearing at one end of our interval. The question is whether they increase or decrease with x_2. To settle this question we notice that, x_1, x_2 being a pair of values for which $y_n \neq 0$, the function y_{n+1}/y_n must, according to what we have proved above, increase when x_1 or x_2 or both increase. If then x_1', x_2' is a pair of values for which y_{n+1} vanishes, and if starting from these values, we wish to change x_1, x_2 so that y_{n+1} retains the value zero, we must increase one of them and decrease the other; *i. e.* if we regard y_{n+1} as a function of x_1, its roots decrease as x_2 increases.

Coming back now to the case in which there are more than two parameters, we will assign to all but two of them fixed values. Then by applying the result just obtained we get the theorem:

If $y_n(x_1, x_2 \ldots)$ be regarded as a function of the variable x_i, the roots of y_n will decrease with the increase of the other parameters.

4. Theorems concerning Sturmian Solutions. We have seen that, if y_n denote a Sturmian solution in a certain interval, then y_n has no multiple roots in this interval and that y_n/y_{n-1} increases continually from $-\infty$ to $+\infty$ in any interval delimited by two consecutive roots of y_{n-1}. Similarly, it is seen at once that $(y_n + H_n y_{n-1})/y_{n-1}$ will increase continually from $-\infty$ to $+\infty$ in this same interval provided that

$$\frac{d}{dx} H_n \geqq 0.$$

A similar extension can of course be given to the last theorem of §3. From the foregoing we can now deduce the following theorems.

THEOREM 1. *A variation will be lost in the sequence $y_0, y_1, \ldots y_n$ whenever y_n vanishes.*

THEOREM 2. *The roots of y_n and y_{n-1} separate each other*, or more generally *the roots of $y_n + H_n y_{n-1}$ and y_{n-1} separate each other provided that $H_n' \geqq 0$.*

For, since y_n/y_{n-1} increases continually from $-\infty$ to $+\infty$ in any interval delimited by two consecutive roots of y_{n-1}, the function y_n, which has no multiple roots, will vanish once and only once when x increases through this interval.

Theorem 3. *If $y_n^{(1)}$ and $y_n^{(2)}$ denote any two linearly independent solutions of* (1′), *then if* $\left(\frac{y_1^{(1)}}{y_0^{(1)}}\right)' > 0$ *and* $\left(\frac{y_1^{(2)}}{y_0^{(2)}}\right)' > 0$, *the roots of $y_n^{(1)}$ and $y_n^{(2)}$ will separate each other.*

From (1′) we have

$$y_n^{(1)}\,y_{n-1}^{(2)} - y_n^{(2)}\,y_{n-1}^{(1)} \equiv y_1^{(1)}\,y_0^{(2)} - y_1^{(2)}\,y_0^{(1)} \equiv C(x)$$

where, by the hypothesis that $y_n^{(1)}$ and $y_n^{(2)}$ are linearly independent, C must be a function of x which does not vanish in the interval (x_0, x_1). Further, since $C(x)$ is continuous, and thus cannot change sign, we may suppose $C(x)$ always positive. Let $\bar{x}_2$ and x_2 denote two consecutive roots of $y_n^{(2)}$. By Theorem 2 this interval contains a single root of $y_{n-1}^{(2)}$; hence, since

$$y_n^{(1)}(x_2)\,y_{n-1}^{(2)}(x_2) = C(x_2) > 0$$

and

$$y_n^{(1)}(\bar{x}_2)\,y_{n-1}^{(2)}(\bar{x}_2) = C(\bar{x}_2) > 0,$$

it follows that $y_n^{(1)}$ vanishes at least once in the interval determined by two consecutive roots of $y_n^{(2)}$. Similarly, $y_n^{(2)}$ vanishes at least once in an interval determined by two consecutive roots of $y^{(1)}$, and hence the theorem.

Let $y_n^{(1)}$ and $y_n^{(2)}$ be any two linearly independent solutions, and let us denote by y_n the solution

$$y_n = zy_n^{(1)} + y_n^{(2)}$$

where z is a parameter. Then we have

$$\frac{y_n}{y_{n-1}} = \frac{zy_n^{(1)} + y_n^{(2)}}{zy_{n-1}^{(1)} + y_{n-1}^{(2)}}$$

and

$$\frac{\partial}{\partial z}\left(\frac{y_n}{y_{n-1}}\right) = \frac{y_n^{(1)}\,y_{n-1}^{(2)} - y_n^{(2)}\,y_{n-1}^{(1)}}{(zy_{n-1}^{(1)} + y_{n-1}^{(2)})^2} = \frac{y_0^{(1)}\,y_0^{(2)}}{(zy_{n-1}^{(1)} + y_{n-1}^{(2)})^2}\left(\frac{y_1^{(1)}}{y_0^{(1)}} - \frac{y_1^{(2)}}{y_0^{(2)}}\right).$$

Thus y_n/y_{n-1} increases continually with z between any two consecutive roots of y_{n-1}, if $y_1^{(1)}/y_0^{(1)} > y_1^{(2)}/y_0^{(2)}$. If z decrease from a very large value to a very small one, y_n/y_{n-1}, starting from a value as near $y_n^{(1)}/y_{n-1}^{(1)}$ as we please, will approach $y_n^{(2)}/y_{n-1}^{(2)}$ as its limit, and the v-points of $y_n \ldots y_0$ will, starting from coincidence with those of $y_n^{(1)} \ldots y_0^{(1)}$, all move in the same sense into coincidence with those of $y_n^{(2)} \ldots y_0^{(2)}$. Since y_n always remains linearly independent of $y_n^{(1)}$ and $y_n^{(2)}$, a v-point of $y_n \cdots y_0$ could not come into coincidence with one of $y_n^{(2)} \ldots . y_0^{(2)}$. If $y_1^{(1)}/y_0^{(1)} < y_1^{(2)}/y_0^{(2)}$ we can reason in the same way, and thus we get

THEOREM 4. *If $y_n^{(1)}$ and $y_n^{(2)}$ denote any two linearly independent solutions of (4) such that $y_0^{(1)}$ and $y_0^{(2)}$ do not vanish, no restrictions as to continually increasing or decreasing being imposed on $G_n(x)$, the v-points of the series $y_n^{(1)} \dots y_0^{(1)}$ and of $y_n^{(2)} \, . \, . \, y_0^{(2)}$ will alternate, i. e. the number of variations in the two sequences will differ at most by unity.*

Theorems 3 and 4 are analogous to well known theorems of Sturm for the linear differential equation of the second order. Theorems 1 and 2 seem to possess no analogon for the differential equation.

The analogon of Sturm's *Theorem of Oscillation* may be stated as follows.

THEOREM 5. *If $G_n(x)$ continually increases with x and is such that $G_n(x_0) \geqq L^2$ and $G_n(x_1) \leqq -M^2$ $(n = 0, 1, 2, \dots)$, and if L^2 and M^2 are sufficiently large, there is one and only one value of $x (x_0 \leqq x \leqq x_1)$ such that 1° The number of variations in the sequence $y_n, y_{n-1}, \dots y_1$ is equal to an assigned integer r (r not greater than $n - 1$) and such that*

2°
$$\frac{y_1(x)}{y_0(x)} = a$$

$$\frac{y_n(x)}{y_{n-1}(x)} = a'$$

a and a' being any assigned real constants.

The proof is immediate. We note that, if in the equation

$$\bar{y}_{n+1} + C\bar{y}_n + \bar{y}_{n-1} = 0$$

we take C as a sufficiently large positive constant, the sequence $\bar{y}_n, \bar{y}_{n-1}, \dots \bar{y}_1$ will present $n - 1$ variations of sign and if C is a negative constant of sufficiently large absolute value, this sequence will present no variations of sign. Thus if L^2 is sufficiently large, the sequence $y_1, y_2, \, . \, . \, y_n$ will present $n - 1$ variations of sign. As x increases from x_0 towards x_1 G will decrease and a variation will be lost every time y_n vanishes; further since the roots of y_{n-1} separate those of y_n and in an interval delimited by two successive roots of y_{n-1}, y_n/y_{n-1} increases continually from $-\infty$ to $+\infty$ with the increase of x, y_n/y_{n-1} passes in every such interval through any given value, as β, once and only once. This proves the theorem.

5. Some Applications. As examples of sequences which fall at once under the conditions of the preceding sections we will cite first the

Zonal Harmonics:

$$P_0(x),\ P_1(x),\ \ldots\ P_n(x)\ ;\quad \text{where } P_0 = 1 \text{ and } P_1 = x,$$

while the recurrent relation is

$$(n + 1)P_{n+1} = (2n + 1)\,xP_n - nP_{n-1}.$$

Since L_n and N_n do not depend on x, the conclusions of §3 are valid and we have, the interval (x_0, x_1) extending from $-\infty$ to $+\infty$,

1° The sequence $P_n \ldots P_0$ is *Sturmian*;

2° The roots of P_n and P_{n-1} separate each other;

3° P_n has no multiple roots.

From 1° it follows that all the roots of P_n are real.*

Lommel's Function: $R^{n,m}(x)$, defined as follows,

$$J_n(x) = R^{n,m-1}(x)\,J_{n+m-1}(x) - R^{n,m-2}(x)\,J_{n+m}(x),$$

where the J's denote Bessel functions, affords a second example.

Here $R^{n,0}(x) = 1$ and $R^{n,1}(x) = 2(n + 1)/x$, the recurrent relation being†

$$(10) \qquad R^{n,m+1}(x) = 2\frac{(n + m + 1)}{x}R^{n,m}(x) - R^{n,m-1}(x),$$

so that, if $n > -1$, the conditions of §3‡ are fulfilled and $R^{n,0}, R^{n,1}, \ldots R^{n,m}$ form a Sturmian sequence. The roots of $R^{n,m}$ and $R^{n,m-1}$ are all real and alternate and increase with the parameter n.

Even if $n < -1$, if $n + m < 0$, the conditions of §3 and the conclusions from them hold.

The conditions at the end of §3 also have place for (10) if $n + m < 0$ and x be changed to $-x$, so that, *if n be increased, the roots of $R^{n,m}(-x) = 0$ decrease; i. e.* the roots of $R^{n,m}(x) = 0$ increase with n.

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* These theorems may be obtained by making use of the relation

$$(x^2 - 1)P_n' = n\,x\,P_n - n\,P_{n-1}$$

which shows that, if $|x| < 1$, $P_n' P_{n-1} > 0$, when $P_n = 0$ (conditions on p. [57]). *Cf.* Weber's *Algebra*, vol. 1, p. 277.

† *Cf.* Graf and Gubler's *Einleitung in die Theorie der Besselschen Functionen*, vol. 2, p. 102.

‡ The inequalities 1° and 2° of p. 62 are here reversed, but Theorems 1, . . . 5 still subsist if we replace the word *lost* in Theorem 1 by *gained* and reverse the inequalities in Theorem 3.

As a final example we may consider the partial numerators and denominators of the convergents of the continued fraction

$$F(x) = \alpha_1 x + \beta_1 - \frac{\gamma}{\alpha_2 x + \beta_2 -} \, \frac{\gamma_2}{\alpha_3 x + \beta_3 -}$$

when the α's are all positive or all negative and the γ's are positive functions of the index n. Denoting the partial numerators and denominators by y_n and $\bar{y}_n$ respectively we have

$$\begin{aligned} y_n &= (\alpha_n x + \beta_n) y_{n-1} - \gamma_n y_{n-2}, \\ \bar{y}_n &= (\alpha_n x + \beta_n) \bar{y}_{n-1} - \gamma_n \bar{y}_{n-2}, \end{aligned}$$

$$y_0 = 1, \qquad y_1 = \alpha_1 x + \beta,$$

$$\bar{y}_0 = 0, \qquad \bar{y}_1 = 1, \qquad \bar{y}_2 = \alpha_2 x + \beta_2.$$

Thus the conditions of §3 are fulfilled and $y_n, y_{n-1}, \ldots\ldots y_0$ (or $\bar{y}_n, \bar{y}_{n-1}, \ldots\ldots \bar{y}_1$) form a Sturmian sequence, all the roots of $y_n(\bar{y}_n)$ are real and are separated by those of $y_{n-1}(\bar{y}_{n-1})$.* From the relation

$$y_n \bar{y}_{n-1} - \bar{y}_n y_{n-1} = \prod_2^n \gamma_n$$

we also infer that the roots of y_n and $\bar{y}_n$ separate each other, *i. e.* the poles and zeros of $y_n/\bar{y}_n$ alternate.

By §3 we conclude that the absolute values of the roots of $y_n(x) = 0$ ($\bar{y}_n(x) = 0$) decrease with $\alpha_r (r \leqq n)$ if the α's are positive, and increase with the α's if the α's are negative.

The foregoing is the simplest type of continued fraction which satisfies the conditions of §§3 and 4.

The general type suggested by §3 is

$$\phi_0(x) - \frac{\gamma_1}{\phi_1(x) -} \, \frac{\gamma_2}{\phi_2(x) - \ldots}$$

where $\phi_n'(x) > 0$.

6. The Limiting Case of the Differential Equation. To show that under certain restrictions, to be hereafter given, the difference equation goes over into a differential equation as its limiting form we shall need the following

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* This result has been established by Sylvester, *Philosophical Transactions*, vol. 14, part I.

Lemma. If $f(x)$ denote a single valued function of the real variable x possessing finite first and second derivatives in the interval (x_0, x_1) then[*]

$$\lim_{\delta=0} \frac{f(x+2\delta) - 2f(x+\delta) + f(x)}{\delta^2} = f''(x), \qquad x_0 < x < x_1.$$

Consider now the difference equation,

$$f_{n+2} + G_n f_{n+1} + f_n = 0 \tag{11}$$

and let

$$f_n = f(a + n\delta), \qquad n\delta = x - a,$$

then (11) may be written

$$\frac{f_{n+2} - 2f_{n+1} + f_n}{\delta^2} = -\left(\frac{2 + G_n}{\delta^2}\right) f_{n+1} \tag{12}$$

If then we suppose that

$$\lim_{\delta=0} \frac{2 + G_n}{\delta^2} = G(x)$$

equation (11) will go over into the differential equation:

$$f''(x) = - G(x) f(x).$$

Conversely, if we have a differential equation in the form

$$y'' = G(x)\, y \tag{13}$$

the Cauchy-Lipschitz existence theorem for solutions of differential equations[†] shows that, if $G(x)$ is continuous (not necessarily analytic) in the neighborhood of a, we can find a solution of (2) having the arbitrary boundary values

$$y(a) = c_1, \qquad y'(a) = c_2 \tag{14}$$

by regarding (13) as the limiting form of a difference equation. In the proof as usually presented (13) is replaced by the simultaneous pair of equations of the first order

$$p' = - G(x)\, y$$
$$y' = p$$

which are regarded as the limit of the difference equations

$$\begin{aligned} p_{n+1} - p_n &= - G(a + n\delta)\, y_{n+1}\delta, \\ y_{n+1} - y_n &= p_n \delta. \end{aligned} \tag{15}$$

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[*] *Cf.*, for example, Harnack, *Elemente der Differential- und Integralrechnung*, p. 56.

[†] See for instance Picard, *Traité d'analyse*, vol. 2, p. 291.

Here we must let

$$y_0 = c_1, \qquad p_0 = c_2.$$

By means of the two equations of (14), all y's and p's of higher order can be computed, and the Cauchy proof shows that

$$\lim_{n=\infty} y_n = y(x), \qquad \lim_{n=\infty} p_n = y'(x)$$

where $y(x)$ is a solution of (13).

The upper limit for $n\delta = |x - a|$ postulated in the classic proof is in general much too small. Picard and Painlevé* have shown that the Cauchy solution will be convergent throughout the whole interval $b_0 b_1$ ($b_0 < a < b_1$), in which $y(x)$ as well as $G(x)$ are continuous and that the convergence is uniform over every tract within this interval. Moreover, since we are dealing with a linear equation, the interval throughout which y is continuous will be at least as great as the interval throughout which G is continuous.†

It is well known‡ that no solution of (13) can have a multiple root at a point of the interval just considered. From this it follows that there cannot be two distinct solutions of (13) satisfying conditions (14), since their difference would have a multiple root at a. On the other hand it is clear that a solution of (2) which is not identically zero cannot have an infinite number of roots in a finite interval within and at the extremities of which G is continuous. For, at the points of condensation of such roots, y' as well as y would be zero.

If we now suppose that $y(a)$, $y'(a)$, and $G(x)$ are continuous functions of a parameter λ ($\lambda_0 < \lambda < \lambda_1$) then y_n and p_n are continuous functions of this parameter and, by the uniformity of the convergence, their limits $y(x)$ and $y'(x)$ are continuous functions of (x, λ).

From the method in which y_n and p_n must be computed by means of (14) it is evident that the result will be the same if we eliminate p between the two equations of (14), thus getting

$$(15) \qquad y_{n+2} - 2y_{n+1} + y_n = - G(a + n\delta)\, y_{n+1}\, \delta^2.$$

If now we plot the ordinates y_m, $m = 0, 1, 2, \ldots n$, as in §1, using for the corresponding abscissas $m\delta$, $m = 0, 1, \ldots n$, we shall get a broken line which, if δ be taken sufficiently small, will lie between the two curves

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* Picard, *Comptes Rendus*, vol. 128 (1899), p. 1363.
Painlevé, *Bull. de la Soc. Math. de France*, vol. 27 (1899), p. 150.

† *Cf.* Peano, *Math. Annalen*, vol. 32 (1888), p. 450.

‡ *Cf.* Sturm, *l. c.* p. 109.

$$\left.\begin{array}{l} y = y(x) + \epsilon \\ y = y(x) - \epsilon \end{array}\right\} \qquad (\epsilon \text{ arbitrarily small})$$

and

where x is supposed to be any point of an interval lying within the interval in which $y(x)$ is continuous.

Equation (15) belongs to the type of difference equations considered in §3, and it was then shown (*cf.* (8)) that if either y_1/y_0 or $G(\lambda)$ increases with λ, the other not decreasing, or both are functions of the parameter λ which continually increase as λ increases $y_n(\lambda)/y_{n-1}(\lambda)$ increases continually with λ in any interval delimited by two successive roots of $y_n(\lambda) = 0$, and that the same is true of $\dfrac{y_n(\lambda) - y_{n-1}(\lambda)}{y_{n-1}(\lambda)\,\delta}$ so that we have the fundamental theorem :

Under the restrictions imposed on $y'(a, \lambda)/y(a, \lambda)$ and $G(x, \lambda)$, namely, that either or both shall continually increase with λ, and neither decrease, the ratio $y'(x, \lambda)/y(x, \lambda)$ will be a continually increasing function of λ in any interval delimited by two successive roots of $y(x, \lambda) = 0$.

In the same way it can be shown that the roots of $y(x, \lambda) = 0$ *will continually increase with* λ.

The Oscillation Theorem for the differential equation can be arrived at in the same way by regarding it as a limiting form of the analogous theorem for the difference equation.

The equation (8) of §3 becomes, if we pass to the limit,

$$y(x, \lambda)\, y'_x(x, \lambda_1) - y(x, \lambda_1)\, y'_x(x, \lambda) =$$
$$\int_a^x [G(x,\lambda) - G(x,\lambda_1)]\, y(x,\lambda)\, y(x,\lambda_1)\, dx + y'_x(a,\lambda_1)\, y(a, \lambda) - y'_x(a,\lambda)\, y(a,\lambda_1).$$

This fundamental equation, which can be at once obtained from the differential equation itself, is the starting point of Sturm and Bôcher* in establishing the Comparison Theorems from which the other Sturmian Theorems are deduced.

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* The Theorems of Oscillation of Sturm and Klein., *Bull. Am. Math. Soc.*, ser. 2, vol. 4 (1898), p. 295.

SPACE OF CONSTANT CURVATURE.

BY FREDERICK S. WOODS.

As is well known, the investigation of the foundations of geometry has led, by three distinct methods, to the recognition of the fact that alongside of the Euclidean geometry there exist other systems, equally self consistent and equally capable of explaining the facts of experience. The first of these methods of research, that of Lobachevsky, Bolyai, and Gauss, proceeds upon the historic lines of Euclid. The second method, that of Klein, starts from Cayley's system of measurement with reference to a fundamental quadric and employs the principles of projective geometry. The third method, that of Riemann, consists essentially in the study of differential quadratic expressions and extends to space of any dimensions Gauss's idea of the curvature of a surface. The first two of these methods make less demand than does the third on special mathematical knowledge, and elementary expositions of them are readily accessible and widely known ; but, as far as the present writer is aware, no elementary treatment of Riemann's ideas exists. Such a treatment seems desirable, however, not only because of the intrinsic importance of Riemann's methods, but also because his results have been sometimes misunderstood. The term "curvature of space" has led some to infer by a false analogy that Riemann's geometry is possible only on the assumption of a fourth dimension.*

The present article is an attempt to present Riemann's ideas in an elementary form. The basis of the article is Riemann's famous paper "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen."† In addition, the writer has consulted, among others, the following papers :

Dedekind-Weber :	Anmerkungen. Riemann's *Gesammelte Werke*, 1st ed., p. 384 ; 2nd ed., p. 405.
Helmholtz :	Ueber die thatsächlichen Grundlagen der Geometrie (1866). *Wissenschaftliche Abhandlungen*, vol. 2, no. 77, p. 610.
	Ueber die Thatsachen, die der Geometrie zum Grunde liegen (1868). *Ibid.*, no. 78, p. 618.
Beltrami :	Teoria fondamentale degli spazii di curvatura constante. *Annali di mat.*, ser. 2, vol. 2 (1868), p. 232.

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* This error has recently been repeated in an article by Professor E. S. Crawley on "Geometry : Ancient and Modern," published in the *Popular Science Monthly* for Jan., 1901, p. 257.

† *Gesammelte Werke*, 1st ed., p. 254 ; 2nd ed., p. 272.

Saggio di interpretazione della geometria non-Euclidea. *Giornale di mat.*, vol. 6 (1868), p. 284.

Newcomb: Elementary theorems relating to the geometry of a space of three dimensions and of uniform positive curvature . . . . *Crelle*, vol. 83 (1877), p. 293.

Killing: Ueber zwei Raumformen mit constanter positiver Krümmung; *Crelle*, vol. 86 (1879), p. 72.

Die Rechnung in den Nicht-Euklidischen Raumformen. *Crelle*, vol. 89 (1880), p. 265.

Ueber die Clifford-Klein'schen Raumformen. *Math. Annalen*, vol. 39 (1891), p. 257.

Einführung in die Grundlagen der Geometrie. Padeborn, 1893.

Schur: Ueber die Deformation der Raüme constanten Riemann'schen Krümmungsmasses. *Math. Annalen*, vol. 27 (1886), p. 163.

Ueber den Zusammenhang der Raüme constanten Riemann'schen Krümmungsmasses mit den projectiven Raümen. *Ibid.*, vol. 27 (1886), p. 537.

Klein: *Autographirte Vorlesungen über Nicht-Euklidische Geometrie*. Göttingen, 1892.

Zur Nicht-Euklidischen Geometrie. *Math. Annalen*, vol. 37 (1890), p. 544.

Lectures on Mathematics, Lecture 11, p. 85, New York, 1894.

Lie-Engel: *Theorie der Transformationsgruppen*, vol. 3, p. 393 ff. Leipzig, 1893.

Bianchi: *Differentialgeometrie*, German translation by Max Lukat, p. 563 ff.

1. The First Two Hypotheses. Riemann's investigations are concerned with the properties of extents (*Mannigfaltigkeiten*) of any number of dimensions, in which an element may be determined by means of coordinates. Such extents are now commonly called "spaces of n dimensions"; but we shall restrict the word "space", in this article, to an extent of three dimensions, while "surface" shall mean an extent of two dimensions, and "line" an extent of one dimension. We have then two problems. First, we have to develop a system of geometry from assumed hypotheses and, secondly, we have to bring our results to the test of experience and inquire how far they are in accord with natural phenomena.

These two problems need to be sharply distinguished. The first is theoretical and exact; the second is empirical and subject to all the uncertainty which belongs to physical observations. In attacking the first problem, the investigator is free to choose his hypotheses as he will, provided only they are not contradictory, and to draw what conclusions he may from them. We shall accordingly make the two following hypotheses which we will first state and afterwards discuss.

FIRST HYPOTHESIS. *The space to be considered shall be a continuum of three dimensions, in which a point may be determined by three independent real quantities*, x_1, x_2, x_3.

SECOND HYPOTHESIS. *The length of a line shall be determined by means of a line element given by the equation*

$$ds = \sqrt{\Sigma a_{ik}\, dx_i\, dx_k}\,, \qquad (a_{ki} = a_{ik}\,;\ i, k = 1, 2, 3)$$

where the a_{ik} are functions of x_1, x_2, x_3, which together with their differential coefficients of the first four orders are finite and continuous, and where the expression under the radical sign is positive for all real values of x_1, x_2, x_3 and dx_1, dx_2, dx_3, provided that x_1, x_2, x_3 correspond to points of the space and that dx_1, dx_2, dx_3 are not all zero.

Discussion of the First Hypothesis. In a purely analytic sense, a "point" is a set of three values (x_1, x_2, x_3), and " space " is a collection, or assemblage, of such points, which together form a continuum of three dimensions. By the requirement that space be a continuum, the following two properties are asserted of space. First, if (a_1, a_2, a_3) is any point of space, then $(a_1 + h_1, a_2 + h_2, a + h_3)$ is also a point of space, whatever may be the values of h_1, h_2, and h_3, provided that no one of them exceeds numerically a certain value h, depending in general on the point (a_1, a_2, a_3). Secondly, if (a_1, a_2, a_3) and (b_1, b_2, b_3) are any two points of space, it is possible to connect them by a continuous curve lying in the space.

Here it is immaterial whether x_1, x_2, and x_3 be allowed to take all possible values, or whether they be subject to certain restrictions, provided only the above two conditions be met.

We assume for the present that to every point of space, or at least to every point of the portion of space which we are studying, corresponds one set of coordinates (x_1, x_2, x_3), and conversely. We can later proceed to all portions of space, in case this condition is not everywhere fulfilled, by the usual methods of analysis.

Our space is assumed to be three dimensional. By this it is not meant to deny the possibility of the space's lying in an extent of higher dimensions, but simply to assert that the discussion has nothing whatever to do with these dimensions. The validity of the results obtained is neither strengthened nor weakened by the assumption of a fourth dimension.

Discussion of the Second Hypothesis. Within the space thus defined we may pick out at pleasure one dimensional extents or lines. We shall restrict ourselves here to lines which may be expressed analytically by the equations

$$x_1 = f_1(t), \qquad x_2 = f_2(t), \qquad x_3 = f_3(t),$$

where t is an arbitrary parameter, and where, for the values of t which come into consideration, f_1, f_2, and f_3 are continuous functions possessing continuous first derivatives which do not vanish simultaneously. Consider now a portion of such a line corresponding to values of t lying between, and including, the values t_0 and T, and let this portion be broken up into n segments, to the extremities of which correspond the values $t_0, t_1, t_2, \ldots\ldots, t_{n-1}, T$. If now (x_1, x_2, x_3) are the coordinates of the extremity $(t = t_i)$ of any segment, the coordinates of the other extremity $(t = t_{i+1})$ will be, except for infinitesimals of higher order, $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$, where

$$dx_1 = f_1'(t)dt, \quad dx_2 = f_2'(t)dt, \quad dx_3 = f_3'(t)dt, \quad \text{and} \quad dt = t_{i+1} - t_i.$$

We introduce now arbitrarily as the definition of an *element of length* a function

$$\phi(x_1, x_2, x_3;\ dx_1, dx_2, dx_3)$$

which has the following two properties. First, it shall become infinitesimal with dx_1, dx_2, dx_3 and consequently with dt, and, secondly, the sum of the n values of this function computed for the n segments of the line considered shall approach a limit as n is indefinitely increased and at the same time the n quantities $t_{i+1} - t_i$ are made to approach zero; the limit to be independent of the manner in which the extremities of the segments are chosen. *This limit is defined as the length of the portion of the line in question.*

These conditions are met in particular by the assumption of the line element as in the second hypothesis, in accordance with which we have

$$\phi = \sqrt{\Sigma a_{ik}\, dx_i\, dx_k}.$$

The length of the line is then given by the integral

$$l = \int_{t_0}^{T} \sqrt{\Sigma a_{ik} f_i'(t) f_k'(t)}\; dt.$$

This form of the line element has for our present purpose the decided advantage of including as special cases all the forms occurring in Euclidean geometry, which correspond to different systems of coordinates (Cartesian, polar, cylindrical, etc.); but it is by no means the only conceivable one. Thus we might, as pointed out by Riemann, employ the positive fourth root of a homogeneous biquadratic expression in dx_1, dx_2, dx_3. Further, the restriction that the expression $\Sigma a_{ik}\, dx_i\, dx_k$ should be always positive for allowable real values of the quantities involved may be dropped,* but in that case we should have certain real lines with zero or even with imaginary length. Such a geometry would not explain the facts of experience and is therefore not considered here.

2. Geodesic Lines and Surfaces. We proceed now to find the equations of the shortest line between two points; or, in other words, to determine the condition that the integral

$$I = \int_{t_0}^{t_1} \sqrt{\Sigma a_{ik} \frac{dx_i}{dt} \frac{dx_k}{dt}}\, dt$$

shall be a minimum. The Calculus of Variations gives, as a necessary condition that the integral

$$I = \int_{t_0}^{t_1} F(x, y, z, x', y', z')\, dt$$

be a minimum, the Eulerian equations

$$\frac{d}{dt} F_{x'} - F_x = 0, \qquad \frac{d}{dt} F_{y'} - F_y = 0, \qquad \frac{d}{dt} F_{z'} - F_z = 0.$$

In the present case, these become

$$\frac{d}{dt} \frac{1}{\sqrt{R_t}} \left(a_{l1} \frac{dx_1}{dt} + a_{l2} \frac{dx_2}{dt} + a_{l3} \frac{dx_3}{dt} \right) = \frac{1}{2} \frac{1}{\sqrt{R_t}} \sum_{ik} \frac{\partial a_{ik}}{\partial x_l} \frac{dx_i}{dt} \frac{dx_k}{dt},$$

where $l = 1, 2, 3$ and $R_t = \Sigma a_{ik} \frac{dx_i}{dt} \frac{dx_k}{dt}$. The integral I is invariant of the parameter t. We introduce the parameter s defined by the equation

$$s = \int_{t_0}^{t} \sqrt{\Sigma a_{ik} \frac{dx_i}{dt} \frac{dx_k}{dt}}\, dt,$$

i. e. the length s of the portion of the geodesic line between an assumed fixed point and a variable point. Since

$$\int_{t_0}^{t} \sqrt{R_t}\, dt = \int_{s_0}^{s} \sqrt{R_s}\, ds = s,$$

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[*] Compare for example Bianchi's paper: "Sulle forme differenziali quadratiche indefinite," *Atti della R. Accademia dei Lincei*, Memorie, ser. 4, vol. 5 (1888), p. 539, as well as the writer's article: Ueber Pseudominimalflächen, Göttingen 1895, wherein are developed properties of certain surfaces in a space for which the line element is $ds = \sqrt{dx^2 + dy^2 - dz^2}$.

it follows that

$$\Sigma a_{ik} \frac{dx_i}{ds} \frac{dx_k}{ds} = 1.$$

Hence the necessary conditions for a minimum, in terms of the parameter s, are

$$\left.\begin{aligned}
\frac{d}{ds}\left[a_{11}\frac{dx_1}{ds} + a_{12}\frac{dx_2}{ds} + a_{13}\frac{dx_3}{ds}\right] &= \tfrac{1}{2}\sum \frac{\partial a_{ik}}{\partial x_1}\frac{dx_i}{ds}\frac{dx_k}{ds}, \\
\frac{d}{ds}\left[a_{12}\frac{dx_1}{ds} + a_{22}\frac{dx_2}{ds} + a_{23}\frac{dx_3}{ds}\right] &= \tfrac{1}{2}\sum \frac{\partial a_{ik}}{\partial x_2}\frac{dx_i}{ds}\frac{dx_k}{ds}, \\
\frac{d}{ds}\left[a_{13}\frac{dx_1}{ds} + a_{23}\frac{dx_2}{ds} + a_{33}\frac{dx_3}{ds}\right] &= \tfrac{1}{2}\sum \frac{\partial a_{ik}}{\partial x_3}\frac{dx_i}{ds}\frac{dx_k}{ds}.
\end{aligned}\right\} \qquad (A)$$

Conversely, these conditions are sufficient if l is not too great. More precisely: Let (a_1, a_2, a_3) be any point of space, and (x_1, x_2, x_3) any second point such that $|x_i - a_i|$ does not exceed a suitably chosen positive quantity, h. Then the equations (A) admit one and only one solution which passes through the points (a) and (x) and has all of its points lying in the region $|x_i - a_i| < h$, and for the corresponding curve the integral l has a smaller value than for any other curve joining the points (a) and (x).

We take the equations (A) accordingly, as the defining equations of the *geodesic lines*. In accordance with the theory of differential equations, it is always possible to find one and only one solution of these equations which takes on at an arbitrary point (x_1, x_2, x_3) any arbitrarily assigned values (not all zero) of the differential coefficients dx_1/ds, dx_2/ds, dx_3/ds. If these differential coefficients satisfy initially the relation

$$\Sigma a_{ik} \frac{dx_i}{ds} \frac{dx_k}{ds} = 1,$$

this relation will be fulfilled for all values of s.

The geodesic lines which radiate from a point are hence distinguished from each other by the ratios of the values of the differential coefficients and these may consequently be regarded as fixing the direction of each line; the direction being, broadly, that property of a geodesic line which distinguishes it from all others through the same point.

It will be convenient to denote dx_i/ds by ξ_i, and to define the triple consisting of the three direction quantities ξ_1, ξ_2, ξ_3 as the *direction* (ξ_1, ξ_2, ξ_3). These quantities satisfy the relation

$$\Sigma a_{ik}\, \xi_i\, \xi_k = 1. \qquad (B)$$

Our results thus far are these :

THEOREM 1. *A geodesic line is completely determined by means of any point on it and any second point not too far distant from the first.*

THEOREM 2. *A geodesic line is completely determined by means of any point on it and its direction at that point.*

THEOREM 3. *Through any point of space pass ∞^2 geodesic lines, each of which is completely determined by means of its initial direction.*

Take now two geodesic lines extending from the same point with the directions $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ and consider the system of geodesic lines which radiate from their point of intersection with the directions (ξ_1, ξ_2, ξ_3), where

$$\begin{aligned}\xi_1 &= \lambda\alpha_1 + \mu\beta_1,\\ \xi_2 &= \lambda\alpha_2 + \mu\beta_2,\\ \xi_3 &= \lambda\alpha_3 + \mu\beta_3,\end{aligned}$$

and, by virtue of the relation (B), λ and μ are connected by the relation

$$\lambda^2 + \mu^2 + 2\lambda\mu\,\Sigma a_{ik}\,\alpha_i\,\beta_k = 1.$$

Such a system we call a *pencil of geodesic lines*, and the common point the *vertex*. This pencil corresponds to the totality of straight lines in elementary Euclidean geometry, all of which go through a common point and lie in the same plane. Any two lines of a pencil define the pencil. Any line of the pencil is fixed by means of a single parameter, which may be conveniently chosen by introducing the conception of an *angle*. *The angle θ between two intersecting geodesic lines, with the directions (ξ_1, ξ_2, ξ_3) and (ξ_1', ξ_2', ξ_3'), shall be defined by the relation*

$$\cos\theta = \Sigma a_{ik}\,\xi_i\,\xi_k',$$

where the a_{ik} are computed for the point of intersection.

The angle θ is always real, since we are considering only real geodesic lines. This may be most conveniently seen by dropping for the moment the condition that (ξ_1, ξ_2, ξ_3) should satisfy relation (B). We may then interpret these quantities as homogeneous point coordinates in a Euclidean plane. If now (ξ_1', ξ_2', ξ_3') and $(\xi_1'', \xi_2'', \xi_3'')$ are two points of this plane, any point of the line connecting them is given by the formula :

$$\begin{aligned} \xi_1 &= \lambda \xi_1' + \mu \xi_2'', \\ \xi_2 &= \lambda \xi_1' + \mu \xi_2'', \\ \xi_3 &= \lambda \xi_1' + \mu \xi_2''; \end{aligned}$$

and the points in which this line intersects the conic section

$$\Sigma a_{ik} \xi_i \xi_k = 0$$

correspond to values of $\lambda : \mu$ given by the equation :

$$\lambda^2 \Sigma a_{ik} \xi_i' \xi_k' + 2 \lambda \mu \Sigma a_{ik} \xi_i' \xi_k'' + \mu^2 \Sigma a_{ik} \xi_i'' \xi_k'' = 0.$$

But the above conic has no real points of intersection by virtue of our second hypothesis. Therefore the roots of the quadratic in $\lambda : \mu$ are imaginary. Therefore its coefficients satisfy the relation :

$$[\Sigma a_{ik} \xi_i' \xi_k'] [\Sigma a_{ik} \xi_i'' \xi_k''] > [\Sigma a_{ik} \xi_i' \xi_k'']^2.$$

Introducing now the condition (B) we have

$$|\Sigma a_{ik} \xi_i' \xi_k''| < 1.$$

The limiting cases $\Sigma a_{ik} \xi_i' \xi_k'' = \pm 1$ can occur only when

$$\xi_1' : \xi_2' : \xi_3' = \xi_1'' : \xi_2'' : \xi_3''.$$

Since the signs of ξ_1, ξ_2, and ξ_3 may be changed without altering the geodesic line corresponding to them (for such a change of sign may be made by altering the sign of ds, that is, of the direction in which the length is measured), it follows that $\cos \theta$ is ambiguous in sign. This ambiguity is the same as that met with in the angle between straight lines in Euclidean geometry and may be removed by familiar conventions.

If $\cos \theta = 0$; *i. e.* if

$$\Sigma a_{ik} \xi_i \xi_k' = 0$$

the two lines are said to be *perpendicular*.

We may now readily prove the following theorems :

THEOREM 4. *In any pencil of geodesic lines there exists one and only one line perpendicular to any given line of the pencil.*

THEOREM 5. *If a geodesic line through the vertex of a pencil is perpendicular to two lines of the pencil, it is perpendicular to all lines of the pencil.*

THEOREM 6. *Through the vertex of a pencil of geodesic lines passes one and only one geodesic line perpendicular to all lines of the pencil.*

THEOREM 7. *The geodesic lines perpendicular to a given geodesic line at a given point form a pencil.*

Proof of Theorem 4. Let a pencil be given with its vertex at O and let OA and OB be any two lines of the pencil with directions $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ respectively, making the angle ω with each other. Then

$$\cos \omega = \Sigma a_{ik}\, \alpha_i \beta_k.$$

Let OM be any line of the pencil defined by OA and OB; then the direction of OM is given by the equations

$$\xi_i = \lambda \alpha_i + \mu \beta_i,$$

where $$\lambda^2 + \mu^2 + 2\lambda\mu \cos \omega = 1.$$

The angle θ between OM and OA is given by the formula

$$\cos \theta = \Sigma a_{ik}\, \alpha_i (\lambda \alpha_k + \mu \beta_k) = \lambda + \mu \cos \omega.$$

The condition that OM be perpendicular to OA is that $\cos \theta = 0$; hence we have as the necessary and sufficient condition for perpendicularity:

$$\lambda = \pm \cot \omega, \quad \mu = \mp \csc \omega.$$

The direction of OM is thus restricted to a single direction and its opposite, and hence there is one and only one line in the pencil perpendicular to OA.

Proof of Theorem 5. Let OC be a geodesic line through OA perpendicular to the line OA and OB of the pencil. The direction $(\gamma_1, \gamma_2, \gamma_3)$ of OC satisfies, then, the conditions

$$\Sigma a_{ik}\, \alpha_i \gamma_k = 0, \qquad \Sigma a_{ik}\, \beta_i \gamma_k = 0,$$

and consequently the condition

$$\Sigma a_{ik} (\lambda \alpha_i + \mu \beta_i) \gamma_k = 0$$

for all values of λ and μ. Hence OC is perpendicular to every line of the pencil.

Proof of Theorem 6. By the equations

$$\Sigma a_{ik}\, \alpha_i \gamma_k = 0, \qquad \Sigma a_{ik}\, \beta_i \gamma_k = 0,$$

the values of the ratios of $\gamma_1 : \gamma_2 : \gamma_3$ are uniquely determined, and by aid of (B) the values of these quantities are determined except for sign. Hence it is always possible to determine one and only one perpendicular to two intersecting geodesic lines at their point of intersection. By Theorem 5, this line is perpendicular to every line of the pencil.

Proof of Theorem 7. If OC with direction $(\gamma_1, \gamma_2, \gamma_3)$ be given, the condition that a geodesic line through O should be perpendicular to OC is

$$\Sigma a_{ik} \gamma_i \xi_k = 0.$$

If now $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ are two sets of values which satisfy this equation, it will be satisfied by all values of the form

$$\xi_i = \lambda \alpha_i + \mu \beta_i$$

and by no other values. This follows from the fact that the equation is linear in the unknown quantities ξ_i. Hence the geodesic lines through O perpendicular to OC form a pencil.

Geodesic Surface. If we define a pencil by means of two perpendicular geodesic lines OA and OB, and denote as before by θ the angle (properly chosen) between OA and any line of the pencil OM, we have

$$\lambda = \cos\theta, \qquad \mu = \sin\theta;$$

so that the direction of OM is given by

$$\xi_i = \alpha_i \cos\theta + \beta_i \sin\theta,$$

where θ is arbitrary and $\Sigma a_{ik} \alpha_i \beta_k = 0$.

If $P(x_1, x_2, x_3)$ be any point on the line OM and r be the length of this line from O to P, the coordinates of P are determined by integrating the differential equations of the geodesic line, choosing the solution which at the point x_i has the direction ξ_i, and substituting r for s. We have then

$$x_i = f_i(\xi_1, \xi_2, \xi_3, r) = \phi_i(\theta, r),$$

the functions $\phi_i(\theta, r)$ being continuous, together with their partial derivatives of the first and second orders. By giving to θ and r all possible values, all points of all lines belonging to the pencil are determined. These points, dependent upon two parameters, form an extent of two dimensions which we call a *geodesic surface*. *A geodesic surface is then defined as a pencil of geodesic lines.* By the definition, the vertex of the pencil forms a unique point of the surface. Whether or not any point of the surface may play this important rôle is at present undecided. We cannot in fact assume that the geodesic surface contains any geodesic lines of space other than those of the pencil.

3. Curvature. *a. The Curvature of a Surface.* Let there be given any surface

$$x_i = \phi_i(u, v), \qquad (i = 1, 2, 3)$$

where u and v are independent variables and the ϕ_i are single-valued continuous functions with continuous differential coefficients of the first two orders. The line element of any line on the surface is found from the line element of space. There results

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

where

$$E = \Sigma a_{ik} \frac{\partial x_i}{\partial u} \frac{\partial x_k}{\partial u},$$

$$F = \Sigma a_{ik} \frac{\partial x_i}{\partial u} \frac{\partial x_k}{\partial v},$$

$$G = \Sigma a_{ik} \frac{\partial x_i}{\partial v} \frac{\partial x_k}{\partial v}.$$

The properties of the surface belong to two distinct classes. The properties of the first class are those which may be deduced from a study of the line element just written, where E, F, G are treated as functions of u and v, and no reference is had to their dependence upon x_1, x_2, x_3; while those of the second class depend upon the relations between u, v and x_1, x_2, x_3. For example, if the line element of the surface is

$$ds^2 = du^2 + dv^2,$$

the surface may be a plane, a cylinder, or any developable surface, if it lies in Euclidean space, and may be something quite different in a non-Euclidean space, and the properties of the first class are those common to all these surfaces. The distinction between the two classes of properties is often lost sight of in elementary presentations of the theory of surfaces, but it is of fundamental importance to us, since the properties of the first class maintain their significance whatever may be the nature of the space in which the surface lies. We may accordingly use the results of the ordinary theory of surfaces, in so far as they concern these properties. To obtain the properties of the first class, we assume the line element of the surface, and define length, geodesic lines on the surface, and angle in a manner similar to that just employed for space. Thus the equations of the geodesic lines are

$$\frac{d}{ds}\left[E\frac{du}{ds}+F\frac{dv}{ds}\right]=\tfrac{1}{2}\left[\frac{\partial E}{\partial u}\left(\frac{du}{ds}\right)^2+2\frac{\partial F}{\partial u}\frac{du}{ds}\frac{dv}{ds}+\frac{\partial G}{\partial u}\left(\frac{dv}{ds}\right)^2\right]$$

$$\frac{d}{ds}\left[F\frac{du}{ds}+G\frac{dv}{ds}\right]=\tfrac{1}{2}\left[\frac{\partial E}{\partial v}\left(\frac{du}{ds}\right)^2+2\frac{\partial F}{\partial v}\frac{du}{ds}\frac{dv}{ds}+\frac{\partial G}{\partial v}\left(\frac{dv}{ds}\right)^2\right]$$

and the angle between two geodesics is defined by the formula

$$\cos\theta =$$

$$E\left(\frac{du}{ds}\right)'\left(\frac{du}{ds}\right)''+F\left[\left(\frac{du}{ds}\right)'\left(\frac{dv}{ds}\right)''+\left(\frac{dv}{ds}\right)'\left(\frac{du}{ds}\right)''\right]+G\left(\frac{dv}{ds}\right)'\left(\frac{dv}{ds}\right)''.$$

It should be noticed that, if the surface is considered as lying in a given space, the geodesic lines on the surface are not in general geodesic lines of the space, but lengths and angles are the same whether measured on the surface or in space.

The properties which may be deduced for a surface by the discussion of its line element depend largely upon the quantity called by Gauss the measure of curvature, or simply the *curvature*, of the surface. Under that term we understand a quantity K, defined by the relation

$$K=\frac{1}{2\sqrt{EG-F^2}}\left\{\frac{\partial}{\partial u}\left[\frac{F}{E\sqrt{EG-F^2}}\frac{\partial E}{\partial v}-\frac{1}{\sqrt{EG-F^2}}\frac{\partial G}{\partial u}\right]\right.$$
$$\left.+\frac{\partial}{\partial v}\left[\frac{2}{\sqrt{EG-F^2}}\frac{\partial F}{\partial u}-\frac{1}{\sqrt{EG-F^2}}\frac{\partial E}{\partial v}-\frac{F}{E\sqrt{EG-F^2}}\frac{\partial E}{\partial u}\right]\right\}.$$

With the geometric interpretation of the curvature as usually given on the hypothesis that the surface lies in Euclidean space we have nothing to do. For us, the "curvature" means simply the above expression, which is fully determined when the line element of the surface is given, and may be shown to be an invariant of the surface, that is, independent of the coordinates chosen to define the surface. When K is the same for all points of the surface, the surface is said to be one of *constant curvature*. The importance of the curvature lies in the two theorems:

A necessary condition that two portions of surfaces may be brought into point for point correspondence with preservation of distance (and hence of angle) is that they have the same curvature at corresponding points.

If the two portions of surfaces are of constant curvature, the condition is also sufficient.

It follows that on any surface of constant curvature there exists a geometry of geodesic lines which is independent of the portion of the surface considered.

Beltrami* has shown that this geometry assumes different distinctive features according as the constant curvature is positive, negative, or zero.

b. The Curvature of Space. As we have seen, the coordinates of a point on a geodesic surface may be taken as (r, θ). The element of arc of a geodesic surface is then

$$ds^2 = E\,dr^2 + 2F\,dr\,d\theta + G\,d\theta^2,$$

where it is possible to determine more exactly the values of the coefficients. In the first place, if $\theta =$ const., $ds = dr$ for all values of r, no matter what value may have been assigned to θ; hence $E = 1$. Secondly, the lines $r = s$, $\theta =$ const. are geodesic lines in space and hence geodesic lines on the surface. Substituting the values $E = 1$, $\frac{d\theta}{ds} = 0$ in the equations of the geodesic lines we have

$$\frac{\partial F}{\partial r} = 0,$$

and therefore F is a function of θ only. When $r = 0$ the quantities x_1, x_2, and x_3 reduce to the coordinates of the vertex of the generating pencil, no matter what the value of θ. Hence

$$\left(\frac{\partial x_1}{\partial \theta}\right)_{r=0} = 0, \qquad \left(\frac{\partial x_2}{\partial \theta}\right)_{r=0} = 0, \qquad \left(\frac{\partial x_3}{\partial \theta}\right)_{r=0} = 0.$$

From the formula

$$F = \Sigma a_{ik} \frac{\partial x_i}{\partial r} \frac{\partial x_k}{\partial \theta},$$

it follows that $(F)_{r=0} = 0$, and hence, since F is a function of θ only, $F \equiv 0$. The line element of the geodesic surface is therefore

$$ds^2 = dr^2 + G\,d\theta^2.$$

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* Saggio di interpretationi della geometria non-Euclidea, *Giornale di matematiche*, vol. 6, (1868), p. 284, *cf.* also, Bianchi, *Differentialgeometrie*, p. 418 ff.

The expression of the curvature reduces in this case to

$$K = -\frac{1}{\sqrt{G}}\,\frac{\partial^2\sqrt{G}}{\partial r^2}.$$

The curvature is consequently a function not only of r and θ, the coordinates of a point on the surface, but also in general of the coordinates of the vertex O of the generating pencil and of the directions $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3)$ and $(\beta_1, \beta_2, \beta_3)$ which fix the pencil, since the functions $\phi_i(r, \theta)$ depend upon these latter quantities. In particular, if K_0 is the curvature at O, it may be shown that

$$K_0 = -\tfrac{1}{2}\,\frac{\sum\limits_{iklm}(ik,\, lm)_0\,(\boldsymbol{\alpha}_i\beta_k - \boldsymbol{\alpha}_k\beta_i)(\boldsymbol{\alpha}_l\beta_m - \boldsymbol{\alpha}_m\beta_l)}{\sum\limits_{iklm}(a_{il}a_{km} - a_{im}a_{kl})_0\,(\boldsymbol{\alpha}_i\beta_k - \boldsymbol{\alpha}_k\beta_i)(\boldsymbol{\alpha}_l\beta_m - \boldsymbol{\alpha}_m\beta_l)},$$

where

$$(ik,\, lm) =$$

$$\frac{\partial^2 a_{il}}{\partial x_k\partial x_m} + \frac{\partial^2 a_{km}}{\partial x_i\partial x_l} - \frac{\partial^2 a_{im}}{\partial x_k\partial x_l} - \frac{\partial^2 a_{kl}}{\partial x_i\partial x_m} + \tfrac{1}{2}\sum_{st}\left(p_{skm}p_{til} - p_{sim}p_{tkl}\right)\frac{A_{st}}{a},$$

$$p_{\lambda\mu\nu} = \frac{\partial a_{\lambda\mu}}{\partial x_\nu} + \frac{\partial a_{\lambda\nu}}{\partial x_\mu} - \frac{\partial a_{\mu\nu}}{\partial x_\lambda},$$

A_{st} is the minor corresponding to a_{st} in the determinant

$$a = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix},$$

the subscript 0 denotes that the expressions to which it is affixed are computed for the point O, and the summation is to be extended over all combinations of the value-pairs i, k and l, m, for which $i < k$, $l < m$.*

The quantity K_0 is called by Riemann the *measure of curvature of the space* in a given point and in a given surface direction. This curvature varies in general from point to point in space, and also in a given point is different for the different geodesic surfaces which pass through the point. In case K_0 is constant, the space is said to be one of *constant curvature*. *A space of constant curvature is accordingly one in which the curvatures of all geodesic surfaces computed for the vertices of the generating pencils are the same.*

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* *Cf.* Dedekind-Weber, *l. c.* also Schur, *Math. Annalen*, vol. 27 (1886), pp. 542–551. A shorter derivation involving the use of differential invariants and Christoffel's symbolic expressions is given by Bianchi, *l. c.* p. 571 ff.

From the expression for K_0 it follows that the necessary and sufficient condition for a space of constant curvature is that the quantities a_{ik} should satisfy identically the relations

$$(ik, lm) = K_0(a_{il}a_{km} - a_{im}a_{kl}),$$

where K_0 is a constant.

4. The Third Hypothesis. We shall define a *displacement* as *any transformation by which a continuous portion of space is brought into a continuous point for point correspondence either with itself or with some other portion of space in such a manner that the lengths of corresponding portions of lines are the same.*

Let S be any portion of space in which the coordinates of any point P are (x_1, x_2, x_3), and let S' be a portion of space in which the coordinates of a point P' are (x'_1, x'_2, x'_3). Let the line element in S be denoted by

$$ds = \sqrt{\Sigma a_{ik}\, dx_i\, dx_k}\,,$$

and the line element in S' by

$$ds' = \sqrt{\Sigma a'_{ik}\, dx'_i\, dx'_k}\,,$$

where a'_{ik} denotes the value of a_{ik} for (x'_1, x'_2, x'_3). In order that S may be displaced into S', it is necessary and sufficient that

$$ds = ds'$$

by virtue of relations of the form

$$x_i = \phi_i(x'_1, x'_2, x'_3), \qquad (i = 1, 2, 3)$$

where the ϕ_i are continuous functions of (x'_1, x'_2, x'_3), possessing continuous first derivatives, and establishing a one-to-one relation between the points of S and S'.

From this follows

$$a'_{ik} = \sum_{lm} a_{lm} \frac{\partial \phi_l}{\partial x'_i} \frac{\partial \phi_m}{\partial x'_k}.$$

Any displacement transforms geodesic lines into geodesic lines, and geodesic surfaces into geodesic surfaces.

The first part of this theorem follows at once from the definition of geodesic lines. To prove the second part, we notice that

$$\frac{dx_i}{ds} = \sum_l \frac{\partial \phi_i}{\partial x'_l} \frac{dx'_l}{ds'} \frac{ds'}{ds} = \sum_l \frac{\partial \phi_i}{\partial x'_l} \frac{dx'_l}{ds'}.$$

Hence, if (a_1, a_2, a_3) is the direction of a geodesic line at a point O, and (a'_1, a'_2, a'_3) the direction of the corresponding line at O',

$$a_i = \sum_l \left(\frac{\partial \phi_i}{\partial x'_l}\right)_0 a'_l.$$

Since this relation is linear, a pencil of geodesic lines with its vertex at O is transformed into a pencil of geodesics with the vertex at O'.

Under any displacement the angle between two geodesic lines is equal to the corresponding angle between the two corresponding geodesic lines.

For let OA and OB be two geodesic lines intersecting at O with the directions (a_1, a_2, a_3) and $(\beta_1, \beta_2, \beta_3)$ respectively, and let $O'B'$ and $O'A'$ be the corresponding lines with directions (a'_1, a'_2, a'_3) and $(\beta'_1, \beta'_2, \beta'_3)$. Then if θ is the angle between OA and OB and θ' the angle between OA' and OB',

$$\begin{aligned} \cos\theta &= \sum_{ik} a_{ik} a_i \beta_k \\ &= \sum_{lm} \left\{ \sum_{ik} a_{ik} \frac{\partial \phi_i}{\partial x'_l} \frac{\partial \phi_k}{\partial x'_m} \right\} a'_l a'_m \\ &= \sum_{lm} a'_{lm} a'_l a'_m = \cos\theta'. \end{aligned}$$

To remove the ambiguity involved in the determination of an angle from its cosine, and at the same time to define corresponding angles, we may consider first the case in which OB coincides with OA. Then $O'B'$ coincides with $O'A'$ and we may take $\theta = \theta' = 0$. Let now OB change its position continuously, the point O being fixed; θ then varies continuously, attaining a final value which we call the angle AOB. At the same time $O'B'$ changes its position continuously, always corresponding to OB, and θ' varies continuously from zero to a final value $A'O'B$, which is the angle corresponding to AOB. Since at all times $\cos\theta = \cos\theta'$, and at the outset $\theta = \theta'$, it follows that $AOB = A'O'B'$.

The possibility of the reversal of the direction of the angle is not excluded, and it seems unnecessary at this point to introduce a distinction between displacements which do not reverse the direction of the angle and those which do.

The existence of displacements is by no means necessary in a general space. This is to be seen by comparing the similar problem in connection with surfaces. A displacement on a surface may be carried out by bending one portion of the surface upon another portion without tearing or stretching. It is clear that surfaces exhibit widely different possibilities in regard to such bending. It may also be seen that the displacement of a portion of a surface and of the entire surface, are different problems. Take, for example, the right circular cylinder in Euclidean space. If a point on the cylinder is fixed, the cylinder as a whole is immovable, but a restricted portion of the cylinder around the fixed point is easily bent upon itself.

The existence of displacements in space is consequently the subject of a new hypothesis.

THIRD HYPOTHESIS. *If P is any point of space, it shall be possible to displace a restricted portion of space surrounding P upon itself in such a manner that any two geodesic lines through P shall correspond to any other two geodesic lines through P, provided only that the two latter lines make the same angle with each other as do the former lines.*

By such a displacement, the point P remains fixed and any two geodesic pencils with their vertices at P may be brought into point for point correspondence with preservation of distance and angle. Hence (*cf.* §3, *a*), the geodesic surfaces formed by the two pencils must have the same curvature at corresponding points and in particular at P. The quantity K_0 of the previous article must therefore be independent of $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$. Now Schur has proved (*l. c.*) that when the curvature of space is constant at each point it does not change as we pass from point to point. Hence K_0 must also be independent of the coordinates of O. We have therefore the theorem:

In order that the third hypothesis shall be met, it is necessary that the space be one of constant curvature.

In the next paragraph we shall show that in a space of constant curvature the hypothesis is always met and that furthermore there exist displacements by which any point P may be transformed into any other point P'.

5. Normal Form of the Line Element.

Let the line element

$$ds^2 = \Sigma a_{ik}\, dx_i\, dx_k \tag{1}$$

belong to a space of constant curvature k^2, where k may be zero, real, or pure imaginary. It is then always possible by a change of coordinates to reduce the line element to Riemann's normal form

$$ds^2 = \frac{dy_1^2 + dy_2^2 + dy_3^2}{\left[1 + \frac{k^2}{4}(y_1^2 + y_2^2 + y_3^2)\right]^2}. \tag{2}$$

This we will assume without proof.* If now $k = 0$, the form (2) is as simple as possible. Assuming for the present that k is not zero, we shall proceed to find another form for the line element. For that purpose, we shall first make the substitutions

$$y_1 = \frac{2u}{t+1}, \quad y_2 = \frac{2v}{t+1}, \quad y_3 = \frac{2w}{t+1}, \tag{3}$$

where u, v, w, and t are connected by the identity

$$1 + k^2(u^2 + v^2 + w^2) = t^2. \tag{4}$$

We have then

$$ds^2 = \frac{du^2 + dv^2 + dw^2 - \frac{1}{k^2}dt^2}{t^2}. \tag{5}$$

Finally let us introduce x, y, and z defined by the relations

$$\left.\begin{aligned} u &= \frac{1}{k}\tan kx, \\ v &= \frac{1}{k}\sec kx \tan ky, \\ w &= \frac{1}{k}\sec kx \sec ky \tan kz, \\ t &= \sec kx \sec ky \sec kz. \end{aligned}\right\} \tag{6}$$

It is readily seen that, if (u_0, v_0, w_0, t_0) is any set of values satisfying (4), and if (b_1, b_2, b_3) and (x_0, y_0, z_0) are the corresponding sets of values of (y_1, y_2, y_3) from (3) and (x, y, z) from (6) respectively then the points (u, v, w, t) that lie in the neighborhood of (u_0, v_0, w_0, t_0) and satisfy (4) are transformed by (3) and (6) respectively in a one-to-one manner and continuously on *all* the points (y_1, y_2, y_3) and (x, y, z) respectively that lie in the neighborhood of (b_1, b_2, b_3) and (x_0, y_0, z_0).

The line element now takes on the *normal form*

$$ds^2 = \cos^2 kz(\cos^2 ky\, dx^2 + dy^2) + dz^2. \tag{7}$$

If k is pure imaginary, the functions in (6) and (7) may be written as hyperbolic functions and the use of imaginaries obviated. We shall avoid duplication of our work, however, by using (6) and (7) as they stand without regard to the reality of k, but it may be noticed that k always appears in such combinations that the resulting expressions are real.

---

* This form is given by Riemann without proof. It is proved by Dedekind-Weber *l. c.* on the assumption that the geodesic surfaces are of constant curvature. Since this assumption has been shown by Schur (*l. c.*) to be warranted, the proof is valid.

A proof has also been given by Lipschitz in connection with extended investigations on the properties of differential expressions, Crelle, vols. 70, 71, 72 (1869/70).

The differential equations of the geodesic lines (A) become, after a few easy simplifications,

$$\left.\begin{aligned}
\frac{d^2x}{ds^2} &= 2k\tan ky\frac{dx}{ds}\frac{dy}{ds} + 2k\tan kz\frac{dx}{ds}\frac{dz}{ds},\\
\frac{d^2y}{ds^2} &= 2k\tan kz\frac{dy}{ds}\frac{dz}{ds} - k\cos ky\sin ky\left(\frac{dx}{ds}\right)^2,\\
\frac{d^2z}{ds^2} &= -k\cos kz\sin kz\left[\cos^2 ky\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right].
\end{aligned}\right\} \qquad (8)$$

The Coordinates (x, y, z). The coordinates (x, y, z) introduced analytically by equations (3), (4), (6) are analogous to Cartesian coordinates. For, we can see by inspection that all lines of the three types:

$$\begin{array}{llll}
(a) & x = \text{const.}, & y = \text{const.}, & z = s;\\
(b) & x = \text{const.}, & y = s, & z = 0;\\
(c) & x = s, & y = 0, & z = 0:
\end{array}$$

satisfy equations (8) and are therefore geodesic lines. In particular, the three lines $(x = s,\ y = 0,\ z = 0)$, $(x = 0,\ y = s,\ z = 0)$ and $(x = 0,\ y = 0,\ z = s)$ form a set of three mutually perpendicular geodesic lines intersecting at O. We will call them OX, OY and OZ respectively. Consider now the surface $z = 0$. We shall have geodesic lines lying on this surface if we place $z = 0$ and take y and x as solutions of the equations

$$\frac{d^2x}{ds^2} = 2k\tan ky\,\frac{dx}{ds}\frac{dy}{ds},$$

$$\frac{d^2y}{ds^2} = -k\cos ky\sin ky\left(\frac{dx}{ds}\right)^2,$$

which result from placing $z = 0$ in (8). These equations, however, are the same as those which we obtain by applying to the surface $z = 0$, in which the line element is

$$ds^2 = \cos^2 ky\,dx^2 + dy^2$$

the equations of the geodesic lines on any surface, as given in §3, a. That is, the shortest line which can be drawn on the surface $z = 0$ connecting any two points is also a geodesic line in space. Hence the surface $z = 0$ is a geodesic surface, any point of which can be taken as the vertex of a generating pencil of geodesic lines. In particular the surface is the geodesic surface determined by OX and OY and may properly be called the surface XOY. Similar results may be obtained for the surfaces $y = 0$ and $x = 0$.

Let now $(x_0, y_0, 0)$ be any point on XOY. Any geodesic line on XOY through this point has then the direction $(\alpha_1, \beta_1, 0)$, where $\alpha_1^2 \cos^2 k y_0 + \beta_1^2 = 1$. Any line of type (a) passing through $(x_0, y_0, 0)$ has the direction $(0, 0, 1)$. Hence the lines of type (a) are perpendicular to the pencil of geodesic lines on XOY through the point in which (a) meet XOY, and are therefore perpendicular to XOY. Similarly the lines of type (b) lie in XOY and are perpendicular to OX. The only line of type (c) is OX. It follows then that if we measure along OX the distance x_0, from the point thus reached measure along a geodesic line perpendicular to OX the distance y_0, and from the point thus attained measure along a geodesic perpendicular to XOY the distance z_0 we reach a point with coordinates (x_0, y_0, z_0). Thus every point (x_0, y_0, z_0) lying in the neighborhood of the point $(0, 0, 0)$ can be reached along one of the paths just described, and along but one such path. As a consequence of the foregoing we have the theorem:

Through any point P lying in the neighborhood of the point $x = y = z = 0$ one perpendicular to the geodesic surface XOY can be drawn and, if we require the perpendicular to lie wholly in the neighborhood of the above point, but one. From any such point P lying in the geodesic surface XOY one perpendicular to the geodesic line OX can be drawn and, under the same restriction as before, but one.

To sum up then: *The coordinate system (x, y, z) involves three mutually perpendicular geodesic lines OX, OY, OZ intersecting at O, and determining three geodesic surfaces XOY, XOZ, YOZ. The coordinates of a point P are determined by taking a geodesic line through P perpendicular to XOY meeting it at M, and from M drawing a geodesic line in XOY perpendicular to OX meeting it at N. The lengths of the lines ON, NM, MP are then x, y and z respectively.*

For the sake of symmetry the three geodesic lines and the three surfaces are specified. Practically only one surface XOY, one line OX, and the point O are needed.

6. The Equations of the Geodesic Lines and Surfaces. To obtain the equations of the geodesic lines we need to integrate the differential equations (8), having regard to (7), or its equivalent,

$$1 = \cos^2 ky \cos^2 kz \left(\frac{dx}{ds}\right)^2 + \cos^2 ky \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2. \tag{9}$$

This may be done by considering the functions derived from (6),

$$\left.\begin{aligned} \xi &= \frac{u}{t} = \frac{1}{k} \sin kx \cos ky \cos kz, \\ \eta &= \frac{v}{t} = \frac{1}{k} \sin ky \cos kz, \\ \zeta &= \frac{w}{t} = \frac{1}{k} \sin kz. \end{aligned}\right\} \tag{10}$$

By linear combinations of (8) and (9), we may obtain the equations:

$$\left.\begin{aligned} \frac{d^2\xi}{ds^2} + k^2\xi = 0, \\ \frac{d^2\eta}{ds^2} + k^2\eta = 0, \\ \frac{d^2\zeta}{ds^2} + k^2\zeta = 0. \end{aligned}\right\} \tag{11}$$

To verify this it is only necessary to multiply equations (8) and (9) by the undetermined factors λ, μ, ν, ρ, to add, and to compare the results with equations (11), written out in terms of x, y, z and their differential coefficients. The values of the multipliers are then easily found. *Hence any solution of equations* (8) *which satisfies* (9) *satisfies also* (11).

Conversely, equations (8) may be expressed as linear combinations of (11) and (9). *Hence any solution of* (11) *which satisfies* (9) *also satisfies* (8).

The general solution of (11) is

$$\left.\begin{aligned} \xi &= \frac{A_1}{k} \sin ks + B_1 \cos ks, \\ \eta &= \frac{A_2}{k} \sin ks + B_2 \cos ks, \\ \zeta &= \frac{A_3}{k} \sin ks + B_3 \cos ks. \end{aligned}\right\} \tag{12}$$

It is necessary to choose the constants so as to satisfy (9). The direct substitution is difficult, but we may proceed indirectly as follows. We notice first that (4) gives us, as a relation which t must satisfy, the following:

$$\xi^2 + \eta^2 + \zeta^2 = \frac{1}{k^2} - \frac{1}{k^2 t^2} \tag{13}$$

Furthermore, (9) is equivalent to (5), and the latter may be written:

$$\left(\frac{d\xi}{ds}\right)^2 + \left(\frac{d\eta}{ds}\right)^2 + \left(\frac{d\zeta}{ds}\right)^2 = 1 - \frac{1}{k^2 t^4}\left(\frac{dt}{ds}\right)^2. \tag{14}$$

Any solution of (11) is also a solution of

$$\xi\frac{d^2\xi}{ds^2} + \eta\frac{d^2\eta}{ds^2} + \zeta\frac{d^2\zeta}{ds^2} + k^2(\xi^2 + \eta^2 + \zeta^2) = 0,$$

which is the same as

$$\frac{1}{2}\frac{d^2}{ds^2}(\xi^2 + \eta^2 + \zeta^2) - \left[\left(\frac{d\xi}{ds}\right)^2 + \left(\frac{d\eta}{ds}\right)^2 + \left(\frac{d\zeta}{ds}\right)^2\right] + k^2(\xi^2 + \eta^2 + \zeta^2) = 0.$$

Hence we get from (13) and (14), as a necessary condition for t, the equation:

$$\frac{d}{ds}\left(\frac{1}{t}\frac{dt}{ds}\right) = \left(\frac{1}{t}\frac{dt}{ds}\right)^2 + k^2.$$

From this

$$\frac{1}{t}\frac{dt}{ds} = k \tan k(s + a),$$

where a is the constant of integration. No generality is lost by taking $a = 0$, since this amounts to denoting by s the length of the arc measured from a particular point, plus a constant; or simply, as will appear later, to measuring the arc from an arbitrary point. Placing $a = 0$ and integrating the last equation we have

$$t = c \sec ks \tag{15}$$

as a necessary condition for the form of the function t.

It follows from the foregoing that, if the equations (12) are to satisfy (9), they must be consistent with (15). Comparison with (14) gives as necessary relations between the coefficients

$$A_1^2 + A_2^2 + A_3^2 = \frac{1}{c^2} + k^2(B_1^2 + B_2^2 + B_3^2) = 1,$$

$$A_1 B_1 + A_2 B_2 + A_3 B_3 = 0.$$

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Conversely, if the seven constants A, B, and c are merely so chosen as to satisfy these three relations and t is then determined by (15), the functions ξ, η, ζ will satisfy (14) and hence x, y, z will satisfy (9).

By multiplying each of the equations (12) by (15), and placing $A_i c = a_i$, $B_i c = b_i$, we have the theorem:

The equations of any geodesic line may be put into the form

$$\left.\begin{aligned} u &= \frac{a_1}{k}\tan ks + b_1, \\ v &= \frac{a_2}{k}\tan ks + b_2, \\ w &= \frac{a_3}{k}\tan ks + b_3, \\ t &= c\sec ks, \end{aligned}\right\} \tag{16}$$

where the constants are connected by the relations

$$\left.\begin{aligned} a_1^2 + a_2^2 + a_3^2 = 1 + k^2(b_1^2 + b_2^2 + b_3^2) = c^2, \\ a_1 b_1 + a_2 b_2 + a_3 b_3 = 0; \end{aligned}\right\} \tag{17}$$

and, conversely, any equations of the form (16) *where the constants satisfy* (17) *represent a geodesic line, if they are satisfied by any points lying in the space under consideration.*

It is important to know that *to any geodesic line corresponds only one set of the above constants, except that all the signs of a_1, a_2, a_3 and c may be changed, and conversely.* For, a geodesic line is completely determined by a point (x_0, y_0, z_0) on it and a direction (α, β, γ), where the signs of all three of the quantities α, β, γ may be changed without altering the line. Then s_0, the value of s corresponding to (x_0, y_0, z_0), is determined by

$$\tan ks_0 = \alpha \tan kx_0 + \beta \tan ky_0 + \gamma \tan kz_0,$$

which is obtained by differentiating the last of equations (6) and (16). The value of s_0 is unique within restricted portions of space. The value of c is then determined by

$$c \sec ks_0 = \sec kx_0 \sec ky_0 \sec kz_0,$$

and the values of a_1, a_2, a_3 are found by differentiating the first three of equations (6) and (16). These latter values depend linearly on α, β, γ. Finally b_1, b_2, b are found from (6) and (16). In this way all the constants are uniquely determined. The converse is proved by noticing that if the equations (16) are given and if, in conjunction with (6), they determine a line lying in the space under consideration, the direction of this geodesic is uniquely determined at any one of its points, and therefore the geodesic is unique. No curve at all results from the equations, however, unless at least one point (x_0, y_0, z_0) within the space considered satisfies the equations.

The equations of the geodesic line are readily written in terms of any point upon it. For, let u_0, v_0, w_0 correspond to any such point and let s_0 be the corresponding value of s. These values determine b , b_2, b_3 and the first three of equations (16) become

$$\left.\begin{aligned} u &= \frac{a_1}{k}(\tan ks - \tan ks_0) + u_0, \\ v &= \frac{a_2}{k}(\tan ks - \tan ks_0) + v_0, \\ w &= \frac{a_3}{k}(\tan ks - \tan ks_0) + w_0. \end{aligned}\right\} \qquad (19)$$

These are of the form

$$\frac{u - u_0}{c_1} = \frac{v - v_0}{c_2} = \frac{w - w_0}{c_3}.$$

Conversely, any equations of the form

$$\frac{u - u_0}{c_1} = \frac{v - v_0}{c_2} = \frac{w - w_0}{c_3},$$

when u_0, v_0, w_0 correspond to a point (x_0, y_0, z_0) of space, determine a geodesic line.

For, the equations may be written

$$\frac{u - (u_0 + \lambda c_1)}{\rho c_1} = \frac{v - (v_0 + \lambda c_2)}{\rho c_2} = \frac{w - (w_0 + \lambda c_3)}{\rho c_3},$$

[illegible]

$$[illegible]$$

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

Conversely, [illegible]

$$[illegible]$$

[illegible] corresponding [illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

Any geodesic surface is represented by a linear equation in u, v, w ; and, conversely, any linear equation in u, v, w represents a geodesic surface, if it represents anything at all in space.

Any point of a geodesic surface may be taken as the vertex of a generating pencil of geodesic lines.

The forms of the equations of geodesic lines and surfaces are the same as those of straight lines and planes in Euclidean geometry. They have, therefore, in common with straight lines and planes those properties which may be deduced from the forms of the equations. In particular we note the following:

Theorem 8. *Any three points not in the same geodesic line and lying in a restricted portion of space determine a geodesic surface.*

Theorem 9. *If two points on a geodesic surface are connected by a geodesic line, the line lies wholly on the surface.*

Theorem 10. *Any two geodesic surfaces intersect in a geodesic line if they intersect at all.*

7. Displacements. We shall now show that our space satisfies the third hypothesis.

For that purpose it will be convenient to introduce an auxiliary space, which shall be Euclidean in its properties and in which u, v, and w shall be the rectangular Cartesian coordinates of a point. This auxiliary space we shall denote by S', while the space of constant curvature, whose properties we are investigating, we shall call S. The quantities u, v, w have then a double interpretation; on the one hand, in S, they are defined by (6), on the other hand in S' they are ordinary Cartesian coordinates. We thus establish a point for point correspondence between S and S' by which geodesic lines and surfaces in S correspond respectively to straight lines and planes in S'.

We proceed now to ask what corresponds in S' to the geodesic distance between two points in S. For that purpose consider the equation of a geodesic line in the form (16) and let A and B be two points upon it, to which correspond respectively the values s_1, u_1, v_1, w_1 and s_2, u_2, v_2, w_2. Then the geodesic distance between A and B is $s_2 - s_1$. From the equations of the geodesic line, with regard to the relations (17) which connect the constants, we readily compute

$$k^2(u_1u_2 + v_1v_2 + w_1w_2) + 1 = c^2(\tan ks_1 \tan ks_2 + 1)$$

$$k^2(u_1^2 + v_1^2 + w_1^2) + 1 = c^2 \sec^2 ks_1,$$

$$k^2(u_2^2 + v_2^2 + w_2^2) + 1 = c^2 \sec^2 ks_2,$$

whence follows

$$\cos(s_2 - s_1) = \frac{k^2(u_1u_2 + v_1v_2 + w_1w_2) + 1}{\sqrt{k^2(u_1^2 + v_1^2 + w_1^2) + 1}\ \sqrt{k^2(u_2^2 + v_2^2 + w_2^2) + 1}}\ .$$

This formula which we have obtained in S has a good significance in S'. For let us take in S' the quadric surface

$$k^2(u^2 + v^2 + w^2) + 1 = 0$$

as the fundamental quadric in a Cayley system of projective measurement. Then, if M is the distance between two points (u_1, v_1, w_1) and (u_2, v_2, w_2), we have, in accordance with a well known formula of projective geometry,

$$\cos M = \frac{k^2(u\ u_2 + v_1v_2 + w_1w_2) + 1}{\sqrt{k^2(u_1^2 + v_1^2 + w_1^2) + 1}\ \sqrt{k^2(u_2^2 + v_2^2 + w_2^2) + 1}}$$

and hence

$$M = s_2 - s_1.$$

Hence we may say: *If we adopt in S' a system of projective measurement referred to the quadric surface*

$$k^2(u^2 + v^2 + w^2) + 1 = 0,$$

then the geodesic distance between two points in S is equal to the projective distance between the two corresponding points in S'.

Now in order that we may obtain a displacement in S it is necessary and sufficient that we find a continuous one-to-one correspondence between the points of S by means of which geodesic lines go into geodesic lines and the length of any portion of a geodesic line is unchanged. That these conditions are necessary was shown in §4; that they are sufficient may be shown by considering the line element as the length of a geodesic line between two indefinitely near points (x, y, z) and $(x + dx, y + dy, z + dz)$.

To any displacement in S corresponds accordingly a transformation in S' by which straight lines go into straight lines and the projective length of any portion of a straight line is unchanged.

The projective geometry teaches that the transformations with these properties are the collineations of which the quadric surface

$$k^2(u^2 + v^2 + w^2) + 1 = 0$$

is an invariant. Accordingly we may say:

The displacements in S correspond to the collineations in S' of which

$$k^2(u^2 + v^2 + w^2) + 1 = 0$$

is an invariant; and conversely.

These collineations are readily written down. They are

$$\left.\begin{aligned} u' &= \frac{\alpha_1 u + \alpha_2 v + \alpha_3 w + \alpha_4}{\delta_1 u + \delta_2 v + \delta_3 w + \delta_4}, \\ v' &= \frac{\beta_1 u + \beta_2 v + \beta_3 w + \beta_4}{\delta_1 u + \delta_2 v + \delta_3 w + \delta_4}, \\ w' &= \frac{\gamma_1 u + \gamma_2 v + \gamma_3 w + \gamma_4}{\delta_1 u + \delta_2 v + \delta_3 w + \delta_4}, \end{aligned}\right\} \qquad (18)$$

where $|\alpha_1 \beta_2 \gamma_3 \delta_4| \neq 0$, and the sixteen constants are connected by the ten conditions:

$$\left.\begin{aligned} k^2(\alpha_i^2 + \beta_i^2 + \gamma_i^2) + \delta_i^2 &= k^2, \qquad (i = 1, 2, 3) \\ k^2(\alpha_4^2 + \beta_4^2 + \gamma_4^2) + \delta_4^2 &= 1, \\ \alpha_i\alpha_h + \beta_i\beta_h + \gamma_i\gamma_h + \delta_i\delta_h &= 0. \qquad (i, h = 1, 2, 3, 4;\ i \neq h) \end{aligned}\right\} \qquad (19)$$

We now need to show that the displacements (18) satisfy the demands of the third hypothesis. To prove this directly we might show first that the angle between two geodesic lines in S is equal to the angle between the corresponding lines in S', provided the latter angle is measured with reference to the quadric surface

$$k^2(u^2 + v^2 + w^2) + 1 = 0.$$

We might then show directly that the third hypothesis is satisfied in S', and hence in S. But this method would demand extended reckoning. We shall choose an indirect method and shall show first that the third hypothesis is satisfied in a particular point, the origin, and shall then show that it is consequently satisfied in any point of space.

Consider then the origin. The origin is unique in our system of coordinates because there, and there only, the angle between two intersecting geodesic lines in S is equal to the Euclidean angle between the two corresponding straight lines in S'.

To show this, consider a geodesic line in the form (16). If this line goes through the origin with the direction (ξ_1, ξ_2, ξ_3), we have by applying the methods of p. 93,

$$s_0 = 0,\ \ b_1 = b_2 = b_3 = 0,$$
$$a_1 = \xi_1,\ a_2 = \xi_2,\ a_3 = \xi_3,\ c = \pm 1.$$

The equation of §2 for the angle between two such lines becomes

$$\begin{aligned}\cos\theta &= \xi_1\xi_1' + \xi_2\xi_2' + \xi_3\xi_3' \\ &= a_1 a_1' + a_2 a_2' + a_3 a_3'.\end{aligned}$$

The corresponding straight lines in S' are

$$\frac{u}{a_1} = \frac{v}{a_2} = \frac{w}{a_3}$$

and

$$\frac{u}{a_1'} = \frac{v}{a_2'} = \frac{w}{a_3'},$$

whence, if ϕ is the Euclidean angle between them,

$$\begin{aligned}\cos\phi &= a_1 a_1' + a_2 a_2' + a_3 a_3' \\ &= \cos\theta.\end{aligned}$$

With proper conventions as to corresponding angles, we have

$$\phi = \theta.$$

Consider now a displacement in S for which the origin is a fixed point. From (18) and (19), it will correspond in S' to a collineation of the type

$$\left.\begin{aligned} u' &= \alpha_1 u + \alpha_2 v + \alpha_3 w, \\ v' &= \beta_1 u + \beta_2 v + \beta_3 w, \\ w' &= \gamma_1 u + \gamma_2 v + \gamma_3 w, \end{aligned}\right\} \ (R)$$

where

$$\alpha_i^2 + \beta_i^2 + \gamma_i^2 = 1, \qquad (i = 1, 2, 3)$$
$$\alpha_i\alpha_k + \beta_i\beta_k + \gamma_i\gamma_k = 0. \qquad (i, k = 1, 2, 3;\ i \neq k)$$

This is a Euclidean rotation about the origin, or a rotation combined with a reflection on one of the coordinate planes, according as $|\alpha_1\beta_2\gamma_3| = \pm 1$. We shall use the rotation only and shall denote it by (R). Let now OK and

OL be any two geodesic lines in S intersecting at the origin and let OM and ON be any two other lines, such that the angle KOL equals the angle MON. We wish to show that there exists a displacement by which OK becomes OM and OL becomes ON.

In S' we have four corresponding straight lines $O'K'$, $O'L'$, $O'M'$, $O'N'$ such that the Euclidean angle $K'O'L'$ equals the angle $M'O'N'$. Now there exists in S' a rotation (R) by which $O'K'$ may be brought into coincidence with $O'M'$ and $O'L'$ with $O'N'$. This is a well known fact which of course has its analytic proof. Hence we conclude that the required displacement exists in S.

The third hypothesis is accordingly satisfied at the origin.

We shall now show that the hypothesis is satisfied at any point of space. Let P be any point of S and PK, PL, PM, PN four geodesic lines such that the angle KPL equals the angle MPN. We wish to show that there exists a transformation of type (18) by which PK goes into PM and PL into PN. Among the transformations (18) there are ∞^3 transformations by means of which P goes into O. Take any particular one of these, no matter which, and call it T. Let PK, PL, PM and PN become OK_1, OL_1, OM_1 and ON_1 by means of T. Then the angle K_1OL_1 equals the angle M_1ON_1, since all displacements preserve angles. Then there exists, as we have seen, a displacement R_1 by which OK_1 goes into OM_1, and OL_1 into ON_1. Let T^{-1} be the inverse transformation of T. Then, if we carry out in order the transformations T, R_1, and T^{-1}, we have a transformation which transforms PK into PM and PL into PN. Moreover, this transformation is of type (18). This follows geometrically from the fact that all the transformations we have used are collineations of which

$$k^2(u^2 + v^2 + w^2) + 1 = 0$$

is invariant, and (18) is the general form of such a collineation.

Hence *the third hypothesis is satisfied in any space of constant curvature.*

We are accordingly justified in proving general propositions by taking the lines involved in any convenient position. We may, for example, apply to any geodesic line, a result already proved for the line OX and may state the following theorem:

THEOREM 11. *Within a properly restricted portion of a geodesic surface, one and only one geodesic line can be drawn perpendicular to a given geodesic line from a point outside of it.*

Corollary. Within a properly restricted portion of space, one and only one perpendicular can be drawn to a geodesic surface from a point outside of it.

8. Space of Zero Curvature. If $k = 0$, Riemann's form of the line element is

$$ds^2 = dx^2 + dy^2 + dz^2.$$

The coordinates of the direction (ξ_1, ξ_2, ξ_3) of a geodesic line are connected by the relation

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$$

and the angle between two geodesic lines is given by the formula

$$\cos\theta = \xi_1 \xi_1' + \xi_2 \xi_2' + \xi_3 \xi_3'.$$

The differential equations of the geodesic lines are

$$\frac{d^2x}{ds^2} = 0, \qquad \frac{d^2y}{ds^2} = 0, \qquad \frac{d^2z}{ds^2} = 0.$$

The general equations of the geodesic lines are, therefore,

$$\frac{x - b_1}{a_1} = \frac{y - b_2}{a_2} = \frac{z - b_3}{a_3} = s$$

where

$$a_1^2 + a_2^2 + a_3^2 = 1.$$

The equation of the geodesic surfaces is

$$lx + my + nz + p = 0.$$

It is evident that the coordinates x, y and z are equivalent to u, v and w of the previous paragraphs, and that Theorems 1–11 still hold. In addition we may deduce a theorem relating to the angles of a triangle which is peculiar to a space of zero curvature. Before stating the theorem, however, we need to make the conceptions of a triangle and of the angles of a triangle precise.

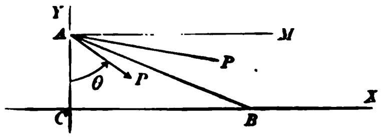
If we select, in the region in which Theorems 1–11 hold, any three points not in the same geodesic line, we may connect them in pairs by geodesic lines and pass a geodesic surface through them. The geodesic lines lie then on this geodesic surface, and the portion of the latter bounded by them we call a *triangle*. The triangle divides the points of the surface not lying on its sides into two classes, *interior* and *exterior* points of the triangle.

To remove the ambiguity inherent in the determination of an angle from the formula above given, we may make the following convention as to the angles of a triangle. If the vertices of the triangle are A, B, and C, the angle at A shall be determined by considering the geodesic lines which radiate from A and enter the triangle ABC. If now a variable line is considered to start from coincidence with AB and take successively the positions of all these lines until it coincides with AC, the angle θ made by the varying line with AB varies continuously from zero to a final value A, which we call the *angle* A of the triangle. We may now state the theorem:

THEOREM 12. *Within a properly restricted region of a space of zero curvature, the sum of the angles of any triangle equals* π.

It will be sufficient to prove this theorem for right triangles, for any triangle may be divided into two right triangles by dropping a perpendicular from any vertex to the opposite side, and if the theorem is true for the two right triangles thus formed, it is evidently true for the original triangle.

Consider the geodesic surface $z = 0$, and a point A of which the (xyz)-coordinates are $(0, p, 0)$, where p is positive. Let A and $C(0, 0, 0)$ determine a geodesic line, on which the direction AC is that of the increasing s. Then the direction of AC is $(\alpha_1 = 0,\ \alpha_2 = -1,\ \alpha_3 = 0)$.



Consider now a moving geodesic turning about A continuously in one direction, thereby fixing the angle θ between AC and AP. By Theorem 1, this line will meet CX at the beginning of its motion. For convenience we take the direction of motion to be such that AP meets CX on the positive side of the origin. Let AM be the first position in which the moving geodesic is perpendicular to AC. If the direction of the increasing s is always taken from A, the direction of AM is $(\beta_1 = 1,\ \beta_2 = 0,\ \beta_3 = 0)$. Then the direction of AP is

$$\xi_1 = \alpha_1 \cos\theta + \beta_1 \sin\theta = \sin\theta,$$
$$\xi_2 = \alpha_2 \cos\theta + \beta_2 \sin\theta = -\cos\theta,$$
$$\xi_3 = \alpha_3 \cos\theta + \beta_3 \sin\theta = 0,$$

and hence the equations of AP are

$$x = s \sin \theta,$$
$$y = -s \cos \theta + p,$$
$$z = 0.$$

Let now AB be a line AP which meets CX, and let B be $(0, q, 0)$, where, by the conventions made, q is positive. If we denote by A the angle CAB, we have from the equations of the line

$$\tan A = \frac{q}{p}.$$

Similarly we may show that

$$\tan B = \frac{p}{q},$$

and hence

$$\cot B = \tan A.$$

Since p and q are both positive, and A and B are both acute angles,

$$B = \frac{\pi}{2} - A.$$

The angle $C = \frac{\pi}{2}$, and hence we have $A + B + C = \pi$, as was to be proved.

As has been repeatedly said, the theorems so far obtained have been shown to be valid only in a region in which there exists a one-to-one relation between the points of space and the coordinates (x, y, z). It is, however, possible to demand that this relation exist for the entire space and all possible values of the coordinates. In this case we have

Euclidean Space.

Here all the theorems so far deduced hold without restriction, and we have in addition two others of interest.

THEOREM 13. *In Euclidean space every geodesic line is of infinite length.*

This follows at once from the form of the equations.

THEOREM 14. *On any geodesic surface in Euclidean space, one and only one geodesic line can be drawn through a given point parallel to a given geodesic line; any other geodesic line through the point intersects the given geodesic.*

We define *parallel lines* as follows: A geodesic line through a given point is said to be parallel to a given geodesic line, when it is the limit approached by a geodesic line through the given point intersecting the given geodesic, as the point of intersection becomes indefinitely remote.

The truth of the theorem then follows from the equations used in proving Theorem (12). We had there

$$\tan A = \frac{q}{p},$$

As the point B recedes indefinitely in either direction, the angle A approaches $\frac{\pi}{2}$, and the parallel line is AM. Any other value of A determines q when p is fixed.

These fourteen theorems form the groundwork of Euclidean geometry.

Euclidean space is not however the only conceivable space of zero curvature. A systematic investigation of all possible cases seems to be yet lacking, although the general lines on which such an investigation would proceed have been marked out by Klein and some of the problems have been attacked by Killing.* We will content ourselves with mentioning three possibilities.

First, we may suppose that to all points with coordinates $(x + n_1 c_1, y + n_2 c_2, z + n_3 c_3)$ where c_1, c_2, c_3 are constants, and n_1, n_2, n_3 are any positive or negative integers, shall correspond the same point of space. Such a conception presents no difficulty, at least analytically. Our space is then represented point for point by the interior of a rectangular parallelopiped in Euclidean space, the lengths of the edges of the parallelopiped being c_1, c_2 and c_3 respectively and the opposite faces corresponding point for point with one another. For any portion of space lying within this parallelopiped, Theorems 1-12 hold, but Theorems 13 and 14 are not always true. For, the lines $(x = \text{const.}, y = \text{const.})$, $(y = \text{const.}, z = \text{const.})$ and $(z = \text{const.}, x = \text{const.})$ are certainly finite in length, and for these lines the theorem of parallels ceases to hold. Moreover triangles may be constructed within the entire space thus defined for which Theorem 12 is not true.

Two other examples of spaces of zero curvature may be derived from the preceding by considering that one or two of the edges of the parallelopiped are indefinite in extent.

9. Space of Constant Negative Curvature. If $k = iR$ where R is a real quantity, the space is one of constant negative curvature, $-1/R^2$.

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* Klein, *Math. Annalen*, vol. 37 (1890) p. 544.
Killing, *Ibid.* vol. 39 (1891) p. 257, and *Grundlagen der Geometrie*, vol. 1, chap. 4.

The element of arc may be written

$$ds^2 = \cosh^2\frac{z}{R}\cosh^2\frac{y}{R}\,dx^2 + \cosh^2\frac{z}{R}\,dy^2 + dz^2,$$

and the functions u, v, w, t are

$$\begin{aligned} u &= R\tanh\frac{x}{R}, \\ v &= R\,\mathrm{sech}\,\frac{x}{R}\tanh\frac{y}{R}, \\ w &= R\,\mathrm{sech}\,\frac{x}{R}\,\mathrm{sech}\,\frac{y}{R}\tanh\frac{z}{R}, \\ t &= \mathrm{sech}\,\frac{x}{R}\,\mathrm{sech}\,\frac{y}{R}\,\mathrm{sech}\,\frac{z}{R}. \end{aligned}$$

To any set of real values of (x, y, z) correspond then one and only one real set of values of u, v, w, t. Since, however, $\mathrm{sech}\,\theta$ is numerically not greater than unity for real values of θ, t is numerically less than unity. That is

$$0 \leqq t^2 \leqq 1$$

and consequently

$$0 \leqq u^2 + v^2 + w^2 \leqq R^2$$

for real values of x, y, and z. Conversely, it is readily proved that to any real values of u, v, w satisfying the above conditions correspond one and only one set of values of x, y, and z.

The angles of a triangle obey the following theorem:

Theorem 12′. *In a properly restricted portion of a space of constant negative curvature, the sum of the angles of any triangle is less than* π.

We may employ the construction and figure of the previous paragraph, but the analytic work will be different. The direction of the line AP must satisfy the relations,

$$\xi_1^2\cosh^2\frac{p}{R} + \xi_2^2 = 1, \qquad \xi_3 = 0.$$

In particular, the direction of AC is $(\alpha_1 = 0,\ \alpha_2 = -1,\ \alpha_3 = 0)$ and of AM $\left(\beta_1 = \mathrm{sech}\,\frac{p}{R},\ \beta_2 = 0,\ \beta_3 = 0\right)$. Hence

$$\xi_1 = \sin\theta\,\mathrm{sech}\,\frac{p}{R}, \quad \xi_2 = -\cos\theta, \quad \xi_3 = 0.$$

By (19), §6 the equations of AP are

$$\left.\begin{aligned} u &= R\tanh\frac{x}{R} = a_1R\left[\tanh\frac{s}{R} - \tanh\frac{s_0}{R}\right], \\ v &= R\operatorname{sech}\frac{x}{R}\tanh\frac{y}{R} = a_2R\left[\tanh\frac{s}{R} - \tanh\frac{s_0}{R}\right] + R\tanh\frac{p}{R}, \\ & z = 0, \end{aligned}\right\}$$

where s_0 is the value of s for the point A. From (16), §6,

$$\operatorname{sech}\frac{p}{R} = c\operatorname{sech}\frac{s_0}{R}.$$

Hence, by differentiating the equations of AP and substituting the coordinates of A,

$$\xi_1 = a_1\operatorname{sech}^2\frac{s_0}{R} = \frac{a_1}{c^2}\operatorname{sech}^2\frac{p}{R},$$

$$\operatorname{sech}^2\frac{p}{R}\,\xi_2 = a_2\operatorname{sech}^2\frac{s_0}{R} = \frac{a_2}{c^2}\operatorname{sech}^2\frac{p}{R}.$$

Therefore $$a_1 = c^2\sin\theta\cosh\frac{p}{R},$$
$$a_2 = -c^2\cos\theta,$$

and the equations of AP may therefore be written

$$u = -\tan\theta\cosh\frac{p}{R}\left(v - R\tanh\frac{p}{R}\right).$$

The line AB is satisfied by placing $x = q$, $y = 0$, $\theta = A$; whence

$$\tan A = \frac{\tanh\frac{q}{R}}{\sinh\frac{p}{R}}.$$

Similarly $$\tan B = \frac{\tanh\frac{p}{R}}{\sinh\frac{q}{R}},$$

and hence $$\tan A\tan B = \frac{1}{\cosh\frac{p}{R}\cosh\frac{q}{R}}.$$

Since p and q are both positive, $A < \frac{\pi}{2}$, $B < \frac{\pi}{2}$, and $\tan A \tan B < 1$; hence

$$A + B < \frac{\pi}{2}$$

and

$$A + B + C < \pi.$$

The theorem therefore holds for right triangles and consequently for all triangles.

We obtain the simplest example of space of constant negative curvature in assuming that a one-to-one correspondence exists between our entire space and the (x, y, z) coordinates. We have then the

SPACE OF LOBACHEVSKY, BOLYAI AND GAUSS.

Within this space, Theorems 1–11 hold universally. In addition we have the following theorems:

THEOREM 13′. *All geodesic lines are infinite in length.*

This follows at once from the equations.

THEOREM 14′. *On any geodesic surface, two geodesic lines can be drawn through a given point parallel to a geodesic line; all other geodesic lines through the given point are separated by the two parallel lines into two sets, one set consisting of lines which intersect the given geodesic line, the other of lines which do not.*

The definition of parallels remains the same as in Euclidean space. The truth of the theorem follows from the equation

$$u = -\tan\theta \cosh\frac{p}{R}\left(v - R\tanh\frac{p}{R}\right)$$

deduced above for a line through $(x = 0, y = p, z = 0)$ in the plane $z = 0$. This line intersects OX, or $v = 0$, in the point

$$u = R\sinh\frac{p}{R}\tan\theta, \qquad v = 0, \qquad w = 0.$$

These values correspond to real, infinite, or imaginary values of x, y, z according as

$$R^2\sinh^2\frac{p}{R}\tan^2\theta <, =, \text{ or } > R^2.$$

Let α be the smallest value of θ for which

$$R^2 \sinh^2 \frac{p}{R} \tan^2 \theta = R^2 ;$$

then
$$\cot \alpha = \sinh \frac{p}{R} = \frac{e^{\frac{p}{R}} - e^{-\frac{p}{R}}}{2}$$

whence
$$e^{\frac{p}{R}} = \cot \tfrac{1}{2}\alpha ,$$

$$\alpha = 2 \cot^{-1} e^{\frac{p}{R}}$$

It is now clear that if $0 \leqq \theta < \alpha$, the geodesic line intersects the line OX, the point of intersection receding indefinitely as θ approaches α; if $\alpha < \theta < \pi - \alpha$, the geodesic line does not intersect OX; if $\pi - \alpha < \theta \leqq \pi$, the geodesic again intersects OX. This establishes the theorem.

We have thus the groundwork of Lobachevsky's geometry.[*]

Klein[†] has pointed out that other examples of space of constant curvature may be found by considering the polyedrons used by Poincaré in his investigations of the Klein groups. The details of the work are yet to be carried out.

10. Space of Constant Positive Curvature. If $k = 1/R$, where R is a real quantity, the space is one of constant positive curvature $1/R^2$. The functions u, v, w, t are then

$$u = R \tan \frac{x}{R} ,$$

$$v = R \sec \frac{x}{R} \tan \frac{y}{R} ,$$

$$w = R \sec \frac{x}{R} \sec \frac{y}{R} \tan \frac{z}{R} ,$$

$$t = \sec \frac{x}{R} \sec \frac{y}{R} \sec \frac{z}{R} ,$$

It appears that u, v, w, t are single valued functions of x, y, z, but the converse is not true.

If we limit ourselves to such a portion of space that a one-to-one relation exists between its points and the coordinates (u, v, w) and (x, y, z), Theorems 1–11 of the preceding paragraphs hold. In addition we have the following theorem:

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[*] *Cf. Zwei geometrische Abhandlungen*, German translation by Engel, Leipzig, 1899.

[†] *Math. Annalen*, vol. 37 (1890), p. 544.

THEOREM 12″. *In a properly restricted portion of a space of constant positive curvature, the sum of the angles of any triangle is greater than π.*

The proof runs exactly parallel with the proof of the corresponding theorem in space of constant negative curvature.

We obtain an example of a space of constant positive curvature, if we assume that a one-to-one correspondence exists between the points of space and all the triples of values (u, v, w). We have then

SIMPLE SPACE OF CONSTANT POSITIVE CURVATURE.

Within this space, the following distinctive theorems are derived from the equations of the geodesic lines:

Every geodesic line is of constant finite length πR.

All geodesic lines intersecting in a point return again to their point of intersection, and have besides no point in common.

There are no parallel lines in this space.

Another example of a space of constant curvature is obtained if we assume that a one-to-one correspondence exists between the points of space and the sets of four values (u, v, w, t) where

$$1 + R^{-2}(u^2 + v^2 + w^2) = t^2.$$

We have then the

DOUBLE SPACE OF CONSTANT POSITIVE CURVATURE.

Within this space the distinctive theorems hold:

Every geodesic line is of constant finite length $2\pi R$.

All geodesic lines intersecting in a point intersect again in a point at a distance πR from the point of intersection and then return to the point of intersection.

Parallel lines do not exist in this space either.

In the double-space the geometry on the geodesic surfaces is identical with that on the surface of a sphere in Euclidean space. In the simple space the geometry on a geodesic surface is that which would be obtained in Euclidean space by the projection of the surface of the sphere from its centre upon a plane.

For further discussion, the reader is referred to the last cited work of Killing.

11. Relation of our Results to the Facts of Experience. There remains yet the problem to enquire how far the foregoing systems of geometry explain the facts of experience. To deal with this question exhaustively would require more space than is at our disposal. We must content ourselves with a mere outline of the treatment.

We do not wish to discuss the metaphysical questions involved. To some minds the space conceptions which underlie Euclid's work, including his theories of parallels and of the infinite extension of space, are *a priori* true, and no others conceivable. With those who hold this opinion we have no argument for we meet them on no common ground. We take the standpoint that our knowledge of external space comes to us solely from experience. A system of geometry is an attempt to explain certain facts of experience and, as long as the system is self-consistent, its objective truth or falsity can be judged only by its success in so doing.

We assert: *Any one of the systems of geometry derived from the hypotheses of this article is, as far as our present knowledge goes, in full accord with all facts of experience, provided suitable values are given to the constants involved in each system.*

Experience is necessarily limited to a finite extent of space. Observation and measurement can never extend beyond the most remote astronomical objects, and this distance can well be very small compared with the full extent of objective space. Hence any geometry which agrees with experience only for a restricted portion of space will serve quite as well as Euclid's geometry for explaining the facts of experience. In fact we are not justified in going outside of such a restricted portion of space. Any assumptions, for example, as to the existence or the nature of parallel lines, the connectivity of space, or the displacement of space as a whole, are only justified in case they produce results which would be visible in an arbitrarily small region of space.

The geodesic lines of any space of constant curvature have for a suitably restricted portion of their length all the properties of the "straight lines" of practical life or of Euclidean geometry. In the endeavor to construct a material line which shall be as nearly as possible "straight," we may proceed by attempting to realize the shortest distance between two points; for example, by free-hand drawing, by stretching a string, or by observing the path of a ray of light. The result is simply a geodesic line by definition. Or, we may look for a line which may be revolved upon itself when two points are fixed. This is also a property of a geodesic line in a space of constant curvature, by virtue of the third hypothesis.

A geodesic surface has the properties of a "plane." The practical testing of a plane surface by the application of a "straight" edge has its full significance in a space of constant curvature.

The practical measurement of length and angle gives no criterion as to the curvature of space. Such measurement consists of the assumption of a unit and the application of the unit to the thing measured. These operations are carried out without hindrance in any space of constant curvature. In fact the difference between a space of constant curvature and Euclidean space is not in the manner in which measurement is practically carried out but in the analytic interpretation of the process. So, more generally, any geometric figure may be carried without distortion from one portion of space of constant curvature to another, so that all properties of figures are independent of the positions of the figures in space.

From these considerations it appears that the groundwork of experimental geometry is unaltered by an assumption as to the curvature of space, provided the curvature is constant. Referring to the theorems of the preceding pages, we see that all spaces of constant curvature agree in the first eleven theorems. A distinction appears first in the twelfth theorem. Here we have apparently a means at hand to determine precisely the curvature of external space by measuring the angles of a triangle. The test however fails, owing to the practical impossibility of exact measurements. All we can discover is that the sum of the angles of a triangle does not differ very much from π. And now it is possible to show that if the sides of a triangle in a space of constant curvature, $\pm 1/R^2$, are sufficiently small compared with R the divergence of the sum of its angles from π is within the limits of the errors of observation.* We can say only this: *If the space of experience is of constant curvature* $\pm 1/R^2$, *R must be very large compared with the distance between any two points lying in that part of space in which measurement is possible.*

To complete our argument, we need to ask whether other crucial theorems may not be developed, by means of which the curvature of space may be practically determined. At present, no such theorem has been found. Euclidean geometry in finite space may be derived from our first twelve theorems, and the geometry in a space of constant curvature agrees, as far as has been shown, with Euclidean geometry within the limits of error of observation.

There are still questions to be solved, however, notably in regard to the connectivity of different spaces of constant curvature. It is only within the last twenty-five years that the possibility has been recognized of the existence of such spaces as that represented by the parallelopiped of §8, and of similar forms with other curvature. Consequently many theorems stated in the earlier papers as universally true are now known to be false, or true only in a restricted portion of space. For example, a space or a surface of zero curvature is not necessarily infinite in extent; two "straight" lines in a space of zero curvature may meet in more than one point; and the sum of the angles of a triangle in a space of zero curvature may exceed π, if the surface of the triangle covers a sufficiently large area. It is possible that a more careful study of these spaces may yet reveal properties which render them unfit to explain the facts of experience, but at present no such properties have been found.

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* See for example the calculation in Lobachevsky's *Zwei geometrische Abhandlungen*, translated by F. Engel, pp. 22–24.

If we are satisfied that the geometries belonging to spaces of constant curvature explain experience, the question naturally arises: *Are these the only systems of geometry which do so?* An answer to this question may be attempted by building up synthetically systems of geometry after the manner of Euclid, but with the assumption of the fewest possible axioms. The most recent example of this mode of procedure is found in the elegant work of Hilbert on the Foundations of Geometry.[*]

If we maintain the analytic methods of this article, the question demands an examination of our three hypotheses. They have been shown to be sufficient to explain experience; the question now is as to their necessity. As to the first hypothesis it is evident that it must be made, if space is to be treated by the ordinary methods of analysis. With the second hypothesis the case is not so clear, since we have assumed the line element arbitrarily. Helmholtz, however, has obtained this line element as a consequence of certain axioms of motion, which include our third hypothesis. Lie[*] has shown that Helmholtz's axioms and methods need a thorough revision, but his results remain essentially unaltered. We are, accordingly, justified in saying that no explanation of spatial phenomena has yet been given without the assumption of axioms which lead necessarily to our three hypotheses.

Massachusetts Institute of Technology,
Boston, December 1901.

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[*] *Grundlagen der Geometrie*, Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmals in Göttingen, Teubner, 1899.

[*] Cf. Lie-Engel: *Theorie der Transformationsgruppen*, vol. 3, pp. 437–543. See also Klein, *Lectures on Mathematics*, l. c.

BRILLIANT POINTS AND LOCI OF BRILLIANT POINTS.[*]

BY W. H. ROEVER.

1. Introduction. When a ray of light meets a polished surface it is reflected according to the following laws:

> The incident ray, the reflected ray, and the normal to the reflecting surface at the point of reflection lie in one plane. This plane is called the plane of reflection.
>
> In the plane of reflection, the incident ray and the reflected ray lie on opposite sides of the normal and on the same side of the tangent.
>
> The incident ray and the reflected ray make equal angles with the normal.

These laws are inferred from experiments in which pencils of light are reflected from polished planes.

The points of reflection of those reflected rays which pass through the pupil of an observer's eye are, for the observer, luminous points, and are called *brilliant points*.[†] The points of reflection of those reflected rays which, when produced backward, pierce the pupil of an observer's eye are *not* luminous, and are sometimes called *virtual* brilliant points. Virtual brilliant points present themselves in the analytic treatment when certain equations are freed from radicals.

Tubular surfaces are envelopes of spheres of constant radii whose centres are situated in a given space curve which is called the axis of the tube. As the radius of a tubular surface approaches zero, its brilliant points approach points in the axis. The axis is the given space curve, and this fact suggests the definition of a brilliant point of a space curve.

It is evident that the position of a brilliant point depends upon the position of the observer's eye, the position and nature of the source of light, and the position of the reflecting surface. Hence when the surface, source or eye moves, or when the surface changes its shape or size, the brilliant points move. The locus of a brilliant point may be a curve, a surface or a region of more dimensions. When it is a curve, it is called a *brilliant curve*, and when a surface, a *brilliant surface*. Owing to the fact that the impression of an image on the retina of the eye remains for some time after the object which has produced it has disappeared or become displaced, it is possible for us to perceive the brilliant curve when the illuminated polished surface is rapidly moved.

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[*] Presented to the American Mathematical Society at its meeting, 22 February, 1902. A communication on this subject has been made to The Academy of Science of St. Louis; *cf. Trans. Acad. Sci. of St. Louis*, vol. x, No 11 (1900), p. lxii.

[†] Alhazen (A. D. 987-1038) determined geometrically the brilliant points of a concave mirror when the source and the recipient are given. *Cf.* Ball, *A Short History of Mathematics*, p. 167.

A striking example of this fact is presented by an illuminated polished rod which is rapidly rotated. When the rod is of small radius and the axis of rotation intersects and is perpendicular to the rod, the brilliant curve lies in a plane. Fig. 3 illustrates this case. The plane of the paper represents the plane of rotation, P_1' and P_2' are the projections of the source of light and of the eye respectively on this plane, and z_1 and z_2 are the distances of the light and of the eye above the plane. The heavy full line represents the visible portion of the brilliant curve. This curve may be seen in a carriage wheel. In this case the source of light is either the sun or a street lamp. Owing to the great distances of the source of light and the eye of the observer from the hub of the wheel (great in comparison with the diameter of the wheel), the brilliant curve looks like a straight line passing through the hub, and on account of the motion of the carriage this line continually changes its position.

Closely packed assemblages of brilliant points look like continuous regions of light when the consecutive brilliant points are so near each other that the eye can not separate them. An example of this is presented by a circular saw which has been polished with emery in a lathe. The consecutive scratches made by the particles of emery are so close that their corresponding brilliant points are too close to be separated by the eye. Each scratch may be considered as being the special position of a variable scratch, and hence the isolated brilliant points are points of a brilliant curve. This curve is illustrated by Fig. 2 and the photographs. In Fig. 2 the plane of the paper below the line AB represents the plane of the saw, P_1' and P_2' are the projections of the source of light and of the eye respectively on this plane, and z_1 and z_2 are the distances of the light and of the eye above the plane. The heavy full line represents the visible portion of the brilliant curve. The photographs also show this curve. In the latter case the source of light is an electric arc and the eye is replaced by the optical centre of the lens of the camera. It will be observed, in the photographs, that near the centre of the saw there is a discontinuity in the brilliant curve. This is due to the fact that, at a small distance from the centre, the concentric circular scratches are replaced by spiral scratches, due to a different mode of polishing.

The most general problem to be discussed in this paper is the following:— *Required to find the locus of the brilliant points of a two parameter family of space curves when the source of light and the eye of an observer are in given fixed positions.* An example under this case is presented by the rapid rotation of the saw above referred to, about an axis in its plane. Another example is furnished by a family of parallel polished wires, as for instance the wires strung on the arms of a telegraph pole.

In what has been said above it has been tacitly assumed that the source of light is a point and that all the rays reflected at the brilliant points belong to a family of right lines each member of which passes through the same point, namely, the pupil of the observer's eye. It might be required to find those points of reflection for which the reflected rays belong to a family of right lines each member of which is normal to a given surface. This given surface would be called the *recipient*. The eye would then be said to be a *point recipient*. A surface to which all the incident rays are normal would be called the *source*. When all the incident rays emanate from a single point, this point would be said to be a *point source*.

In the present paper I shall confine myself to the brilliant points of curves with respect to a point source and a point recipient.

2. Definition of Brilliant Points. Given the space curve

$$F_1(x, y, z) = 0, \qquad F_2(x, y, z) = 0,$$

a family of right lines emanating from the point P_1, (x_1, y_1, z_1) called the *source*, a family of right lines emanating from the point P_2, (x_2, y_2, z_2) called the *recipient*. The point P_0, (x_0, y_0, z_0) is said to be a *brilliant point of the space curve* $F_1 = 0$, $F_2 = 0$ *with respect to the points* P_1 *and* P_2 when the following conditions are fulfilled.

1. The point P_0 is a point of the curve $F_1 = 0$, $F_2 = 0$;
2. The right line $P_0 P_1$, considered as a whole, and the right line $P_0 P_2$, considered as a whole, make equal angles with the line of intersection of their plane with the plane normal to the curve $F_1 = 0$, $F_2 = 0$ at P_0. If the two planes coincide, P_0 shall still be considered as fulfilling this condition.

This is the most natural definition of the brilliant point of a space curve as suggested by that of a surface. Condition 2 is co-extensive with the following condition.

3. The right line $P_0 P_1$, considered as a whole, and the right line $P_0 P_2$, considered as a whole, make equal angles with the right line which is tangent to the curve $F_1 = 0$, $F_2 = 0$ at P_0.

Fig. 1 enables us to see geometrically that Conditions 2 and 3 are coextensive. In the figure, P_0 is the centre of a sphere which is pierced in the points A, A by the tangent at P_0 to the space curve $F_1 = 0$, $F_2 = 0$, in the point B_1 by the right line $P_0 P_1$, in the point B_2 by the right line $P_0 P_2$, and in the point C by the line of intersection of the normal plane and the plane of the lines $P_0 P_1$, $P_0 P_2$. The great circle through C which has A, A for poles is the intersection of the sphere by the plane which is normal to the curve $F_1 = 0$, $F_2 = 0$ at P_0. The great circles AB_1 and AB_2 cut the normal plane in D_1 and D_2 respectively. In the right spherical triangles $B_1 D_1 C$ and $B_2 D_2 C$, angle $D_1 CB_1 =$ angle $D_2 CB_2$. By Condition 2, side $CB_1 =$ side CB_2. Hence the two triangles are equal and arc $AB_1 =$ arc AB_2. But this is Condition 3. If, on the other hand, we have given arc $AB_1 =$ arc AB_2, it follows that arc $CB_1 =$ arc CB_2, and this is Condition 2.

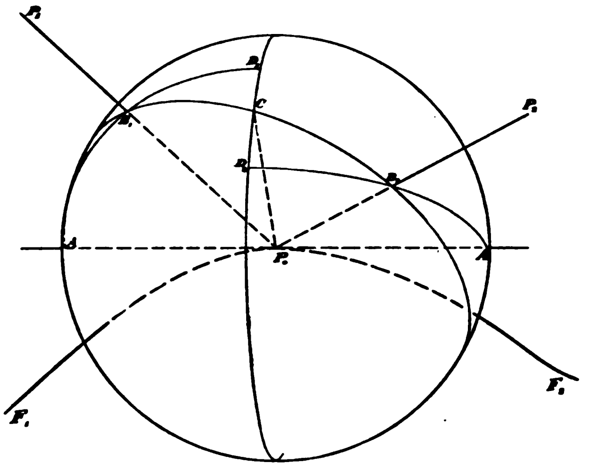

FIG. 1.

Classification of Brilliant Points. Those brilliant points P_0 for which the segments of right lines $P_0 P_1$ and $P_0 P_2$ lie on opposite knappes of a cone of revolution whose axis is the tangent to the space curve at P_0 and whose vertex is P_0, shall be called *actual brilliant points*, and those for which these segments lie on the same knappe shall be called *virtual brilliant points.*

When the semi-angle of this cone is a right angle, an ambiguity arises. In this case P_0 shall be called an actual brilliant point, except when it lies between P_1 and P_2 on the right line P_1P_2. In this exceptional case, it shall be called a virtual brilliant point.

If the curve represented by the equations $F_1 = 0$, $F_2 = 0$ be the axis of a polished wire of small radius, and if one of the points P_1, P_2 be replaced by a source of light and the other by the eye of an observer, then the observer will see the actual brilliant points as luminous points.

3. Analytical Conditions. The deduction of the following conditions will be given in a paper which the author intends to publish in the near future.

The necessary and sufficient condition that the point* $P, (x, y, z)$ *shall be a brilliant point of the space curve* $F_1(x, y, z) = 0$, $F_2(x, y, z) = 0$, *with respect to the two points* $P_1, (x_1, y_1, z_1)$ *and* $P_2, (x_2, y_2, z_2)$ *is*

$$\left.\begin{aligned} F_1(x, y, z) &= 0 \\ F_2(x, y, z) &= 0 \end{aligned}\right\} \qquad (a)$$

$$\begin{aligned} &A_2\left\{B_1\left[\frac{\partial F_1}{\partial x}(x - x_2) + \frac{\partial F_1}{\partial y}(y - y_2) + \frac{\partial F_1}{\partial z}(z - z_2)\right]\right. \\ &\qquad \left. + B_2\left[\frac{\partial F_1}{\partial x}(x - x_1) + \frac{\partial F_1}{\partial y}(y - y_1) + \frac{\partial F_1}{\partial z}(z - z_1)\right]\right\} \\ &- A_1\left\{B_1\left[\frac{\partial F_2}{\partial x}(x - x_2) + \frac{\partial F_2}{\partial y}(y - y_2) + \frac{\partial F_2}{\partial z}(z - z_2)\right]\right. \\ &\qquad \left. + B_2\left[\frac{\partial F_2}{\partial x}(x - x_1) + \frac{\partial F_2}{\partial y}(y - y_1) + \frac{\partial F_2}{\partial z}(z - z_1)\right]\right\} \\ &= 0, \qquad (b) \end{aligned}$$

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* This condition is sufficient, provided that there are no points at which the surfaces $F_1 = 0$, $F_2 = 0$ are tangent and hence

$$\begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{vmatrix} = 0, \quad \begin{vmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial x} \end{vmatrix} = 0, \quad \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix} = 0;$$

and no points at which either $\frac{\partial F_1}{\partial x} = 0$, $\frac{\partial F_1}{\partial y} = 0$, $\frac{\partial F_1}{\partial z} = 0$ or $\frac{\partial F_2}{\partial x} = 0$, $\frac{\partial F_2}{\partial y} = 0$, $\frac{\partial F_2}{\partial z} = 0$.

where

$$A_i = \begin{vmatrix} \frac{\partial F_i}{\partial x} & \frac{\partial F_i}{\partial y} & \frac{\partial F_i}{\partial z} \\ x - x_1 & y - y_1 & z - z_1 \\ x - x_2 & y - y_2 & z - z_2 \end{vmatrix}, \qquad B_i = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ x - x_i & y - y_i & z - z_i \end{vmatrix}, \quad i = 1, 2.$$

By putting, in equations (a), (b),

$$F_1(x, y, z) = F(x, y) = 0, \qquad F_2(x, y, z) = z = 0,$$

and hence

$$\frac{\partial F_2}{\partial x} = 0, \qquad \frac{\partial F_2}{\partial y} = 0, \qquad \frac{\partial F_2}{\partial z} = 1,$$

$$\frac{\partial F_1}{\partial x} = \frac{\partial F}{\partial x}, \qquad \frac{\partial F_1}{\partial y} = \frac{\partial F}{\partial y}, \qquad \frac{\partial F_1}{\partial z} = 0,$$

in which $F(x, y)$, $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ are functions of the variables x and y only, we obtain the following theorem:

The necessary and sufficient condition that the point $P, (x, y)$ *shall be a brilliant point of the plane curve* $F(x, y) = 0$, *with respect to the two points* $P_1, (x_1, y_1, z_1)$ *and* $P_2, (x_2, y_2, z_2)$, *which are not in the plane of the curve is*

$$F(x, y) = 0 \qquad (a')$$

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \left\{ \left[\frac{\partial F}{\partial x}(x - x_1) + \frac{\partial F}{\partial y}(y - y_1)\right] \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x - x_2 & y - y_2 \end{vmatrix} \right.$$

$$\left. + \left[\frac{\partial F}{\partial x}(x - x_2) + \frac{\partial F}{\partial y}(y - y_2)\right] \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x - x_1 & y - y_1 \end{vmatrix} \right\}$$

$$+ z_1^2 \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x - x_2 & y - y_2 \end{vmatrix}^2 - z_2^2 \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x - x_1 & y - y_1 \end{vmatrix}^2 = 0 \qquad (b')$$

Let us denote by a_1 and a_2 the absolute values of z_1 and z_2 respectively and by P_1' and P_2' the projections of the points P_1 and P_2 on the X, Y plane. The condition just stated is the same for the four cases: $(z_1 = a_1, z_2 = a_2)$, $(z_1 = -a_1, z_2 = -a_2)$. $(z_1 = -a_1, z_2 = a_2)$, $(z_1 = a_1, z_2 = -a_2)$. According as we have one of the second two, or one of the first two of these cases, the right line $P_1 P_2$ cuts the right line $P_1' P_2'$ between or not between P_1' and P_2'. Let us denote by P_i the point between P_1' and P_2' and by P_j the other point. The four points P_j, P_1', P_i, P_2' form a harmonic range. The coordinates of P_i and P_j are respectively

$$\left(\frac{x_1 + \frac{a_1}{a_2} x_2}{1 + \frac{a_1}{a_2}}, \frac{y_1 + \frac{a_1}{a_2} y_2}{1 + \frac{a_1}{a_2}}\right), \left(\frac{x_1 - \frac{a_1}{a_2} x_2}{1 - \frac{a_1}{a_2}}, \frac{y_1 - \frac{a_1}{a_2} y_2}{1 - \frac{a_1}{a_2}}\right).$$

Both P_i and P_j satisfy equation (b'), and hence if the curve (a') passes through them, they are brilliant points.

By putting, in equation (b'), $z_1 = 0$ and $z_2 = 0$, we obtain the following theorem:

The necessary and sufficient condition that the point $P, (x, y)$ shall be a brilliant point of the plane curve $F(x, y) = 0$, with respect to the two points $P_1, (x_1, y_1)$ and $P_2, (x_2, y_2)$, which are in the plane of the curve is

$$F(x, y) = 0 \qquad (a'')$$

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \left\{ \left[\frac{\partial F}{\partial x}(x - x_1) + \frac{\partial F}{\partial y}(y - y_1)\right] \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x - x_2 & y - y_2 \end{vmatrix} \right.$$
$$\left. + \left[\frac{\partial F}{\partial x}(x - x_2) + \frac{\partial F}{\partial y}(y - y_2)\right] \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x - x_1 & y - y_1 \end{vmatrix} \right\} = 0 \qquad (b'')$$

The first factor of the left hand member of equation (b'') vanishes for all points of the right line $P_1 P_2$. With the exception of P_1 and P_2, the second factor vanishes for *no* point of this right line, except when the line is normal or tangent to the curve $F = 0$. To show this, substitute for x and y the coordinates of any point of the right line $P_1 P_2$

$$x = \frac{x_1 + k x_2}{1 + k}, \quad y = \frac{y_1 + k y_2}{1 + k}.$$

The second factor of the left hand member of (b'') then becomes

$$-\frac{2k}{(1+k)^2}\left[\frac{\partial F}{\partial x}(x_1-x_2)+\frac{\partial F}{\partial y}(y_1-y_2)\right]\begin{vmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y}\\ x_1-x_2 & y_1-y_2\end{vmatrix}.$$

This expression vanishes when $k=0$ and when $k=\infty$. These are the coordinates of P_1 and P_2. For other values of k the expression cannot vanish unless

either $$\frac{\partial F}{\partial x}(x_1-x_2)+\frac{\partial F}{\partial y}(y_1-y_2)=0 \tag{1}$$

or $$\begin{vmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y}\\ x_1-x_2 & y_1-y_2\end{vmatrix}=0. \tag{2}$$

If the right line P_1P_2 and the curve $F(x, y)=0$ meet, equation (1) is the necessary and sufficient condition that P_1P_2 shall be tangent to $F=0$, and equation (2) is the necessary and sufficient condition that P_1P_2 shall be normal to $F=0$.

4. Loci of Brilliant Points. Suppose we have a family of space curves given by the equations

$$\Phi_1(x, y, z, p_1)=0, \qquad \Phi_2(x, y, z, p_2)=0. \tag{α}$$

in which p_1 and p_2 are two independent parameters. If, after substituting Φ_1 and Φ_2 for F_1 and F_2 in equations (a), (b), we eliminate p_1 and p_2 between these three equations, there results an equation, independent of p_1 and p_2, which is satisfied by the brilliant points of every member of the two parameter family of space curves (α). This equation represents the surface which is the locus of all the brilliant points, *i. e.* the *brilliant surface*, of the given family of space curves with respect to the fixed points P_1 and P_2. When the functions Φ_1 and Φ_2 are of the form

$$\Phi_1(x, y, z, p_1)=F_1(x, y, z)-p_1, \qquad \Phi_2(x, y, z, p_2)=F_2(x, y, z)-p_2,$$

no elimination is necessary, and equation (b), as it stands, represents the brilliant surface. Similarly, if we have a family of plane curves given by the equation

$$\Phi(x, y, p)=0, \tag{α'}$$

and if, after substituting Φ for F in equations (a'), (b'), we eliminate p between these equations, we obtain the equation of the *brilliant curve* of the given family of plane curves (a') with respect to the fixed points P_1 and P_2, which are not in the plane of the curves. When the function Φ is of the form

$$\Phi(x, y, p) = F(x, y) - p,$$

no elimination is necessary, and equation (b'), as it stands, represents the brilliant curve. When both P_1 and P_2 lie in the plane of the family of curves (a'), we obtain the equation of the brilliant curve by eliminating p between equations (a''), (b'') after substituting Φ for F. In this case the brilliant curve consists of two distinct curves, one of which is the right line connecting P_1 and P_2.

THEOREM. *When the source and the recipient are in the plane of a family of curves, the brilliant curve of this family and that of its orthogonal trajectories are identical. The portion of the brilliant curve, with the exception of the right line $P_1 P_2$, which for the one family is the locus of the actual brilliant points, is for the other family the locus of the virtual brilliant points, and vice versa.*

5. Applications. (*a*) *The Saw Curve.* Let it be required to find the equation of the "saw curve" referred to in the introduction. The curve of light seen by an observer and shown in the accompanying photographs is that portion of the brilliant curve which is the locus of the actual brilliant points. In Fig. 2 the locus of the actual brilliant points is represented by a heavy full line and that of the virtual brilliant points by a heavy dashed line. The scratches caused by the particles of emery are concentric circles whose common centre is the centre of the saw, and hence our family of curves is represented by the equation

$$x^2 + y^2 - a^2 = 0, \tag{3}$$

in which a is the variable parameter. $P_1, (x_1, y_1, z_1)$ is the source and P_2, (x_2, y_2, z_2) is the recipient. The derivatives of the left hand member of equation (3) are

$$\frac{\partial F}{\partial x} = 2x, \qquad \frac{\partial F}{\partial y} = 2y.$$

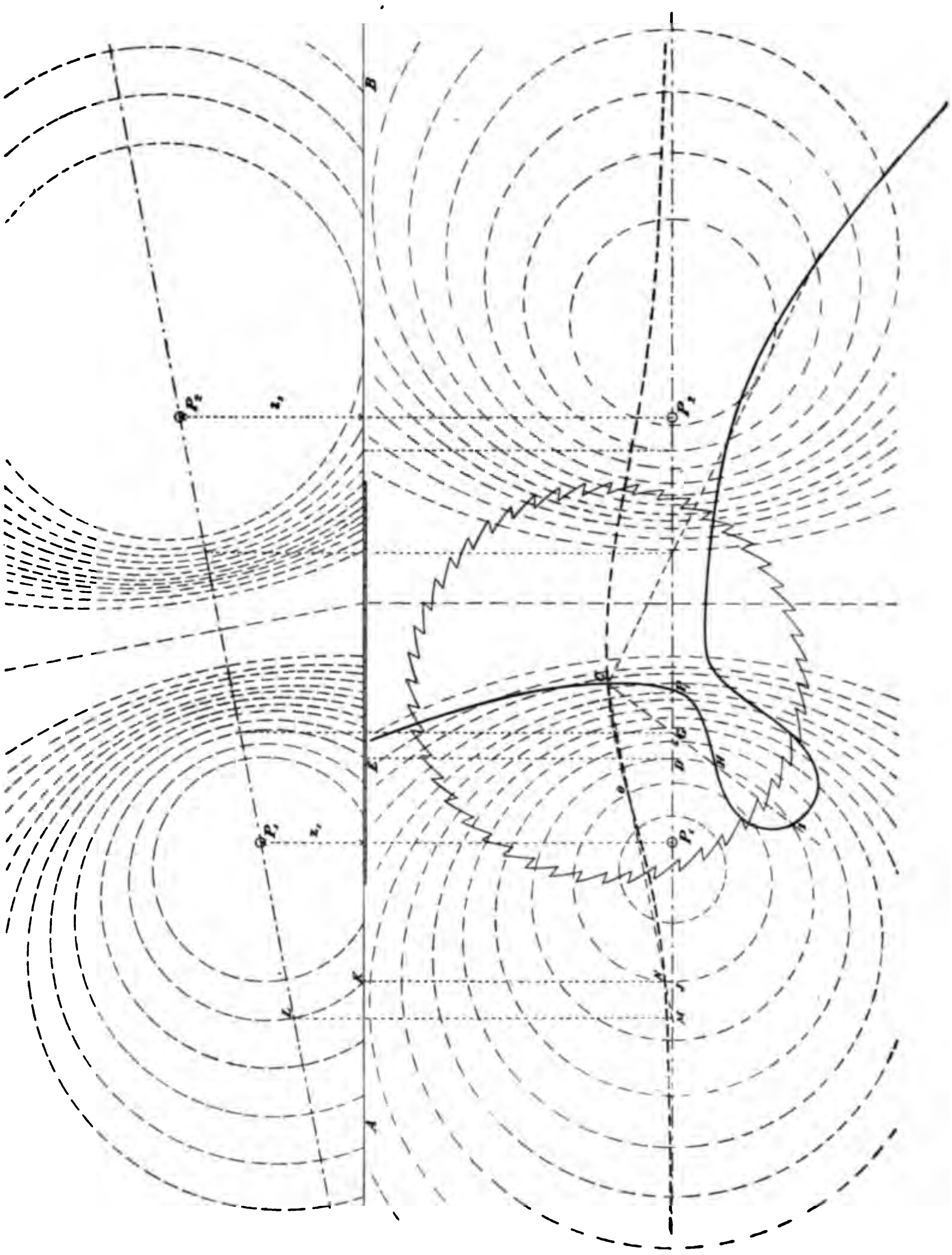

FIG. 2.

Substituting these in (b') we obtain

$$\Big[(x_2 y - y_2 x) + (y_1 x - x_1 y) + (x_1 y_2 - x_2 y_1)\Big]\Big[\{x(x - x_1) + y(y - y_1)\}$$
$$\{x_2 y - y_2 x\} + \{x(x - x_2) + y(y - y_2)\}\{x_1 y - y_1 x\}\Big]$$
$$+ z_1^2(x_2 y - y_2 x)^2 - z_2^2(x_1 y - y_1 x)^2 = 0. \qquad (4)$$

This equation may also be written as follows:

$$\Big[(x_2 - x_1)y - (y_2 - y_1)x + x_1 y_2 - x_2 y_1\Big]\Big[(x_2 + x_1)(y^3 + x^2 y)$$
$$- (y_2 + y_1)(x^3 + y^2 x) + (x_1 y_2 + y_1 x_2)(x^2 - y^2) + 2(y_1 y_2 - x_1 x_2)xy\Big]$$
$$- 2(z_1^2 x_2 y_2 - z_2^2 x_1 y_1)xy + (z_1^2 x_2^2 - z_2^2 x_1^2)y^2 + (z_1^2 y_2^2 - z_2^2 y_1^2)x^2 = 0. \qquad (5)$$

The result may be stated thus: *The brilliant curve of a family of concentric circles, when the source and recipient are not in the plane of the family, is a curve of the fourth degree.* This curve passes through the common centre of the circles (3), and also through the points P_i and P_j, whose coordinates have been given in §3.

When the right line $P_1'P_2'$, which is the projection of P_1P_2 on the plane of the circles, passes through the centre of the circles, $\frac{x_1}{x_2} = \frac{y_1}{y_2}$. For this relation, equation (4) becomes

$$(cy - x)^2\Big[(y_2^2 - y_1^2)(x^2 + y^2) - (y_2 - y_1)(x_1 y_2 + x_2 y_1)x - 2(y_1 y_2)(y_2 - y_1)y$$
$$+ z_1^2 y_2^2 - z_2^2 y_1^2\Big] = 0, \qquad (6)$$

in which $c = \frac{x_1}{y_1} = \frac{x_2}{y_2}$. The first factor of equation (6), when set equal to zero, represents two coincident right lines which pass through the origin and contain P_1' and P_2'. The second factor, when set equal to zero, represents the circle cut by the (X, Y)-plane from the sphere whose equation is

$$\frac{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}{(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2} = k^2, \qquad (7)$$

in which $k = \frac{x_1}{x_2} = \frac{y_1}{y_2}$. This sphere cuts the right line P_1P_2 in points whose coordinates are

$$\frac{x_1 \pm kx_2}{1 \pm k}, \quad \frac{y_1 \pm ky_2}{1 \pm k}, \quad \frac{z_1 \pm kz_2}{1 \pm k}.$$

On account of the value of k, the first two coordinates of one of these points are zero. The result of putting $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = 0$ can be expressed as follows:

When the axis of the saw (i. e. the right line perpendicular to the plane of the saw at its centre) intersects the right line $P_1 P_2$, which connects the source of light, P_1, and the recipient, an observer's eye, P_2, the brilliant curve consists of a straight line and a circle. The straight line passes through the centre of the saw and contains the projections of P_1 and P_2. The circle is the intersection of the plane of the saw by the sphere which has P_1 and P_2 as conjugate points, and passes through the intersection of the axis of the saw and the right line $P_1 P_2$.

When the axis of the saw intersects the right line $P_1 P_2$ in the point P_3 half way between P_1 and P_2, the sphere is replaced by the plane which is perpendicular to $P_1 P_2$ at P_3, and in this case the brilliant curve consists of two right lines which are perpendicular to each other and one of which passes through the centre of the saw. When in this special case the right line $P_1 P_2$ is parallel to the plane of the saw, the two right lines cross each other perpendicularly at the centre of the saw.

The photographs show some of the special cases.

When $z_1 = 0$ and $z_2 = 0$, the curve represented by equation (5) degenerates into the curves represented by the equations*

$$(x_2 - x_1)y - (y_2 - y_1)x + x_1 y_2 - x_2 y_1 = 0 \tag{8}$$

and

$$(x_2 + x_1)(y^3 + x^2 y) - (y_2 + y_1)(x^3 + y^2 x) + (x_1 y_2 + y_1 x_2)(x^2 - y^2) + 2(y_1 y_2 - x_1 x_2)xy = 0. \tag{9}$$

Equations (8), (9) might have been gotten directly from (b''). It is, of course, a physical impossibility to see the curve represented by equation (9).

(*b*) *The Carriage Wheel Curve.* Let it be required to find the equation of the "carriage wheel curve" referred to in the introduction. In Fig. 3

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* Equation (9) is the equation obtained by Lieutenant Hamilton, ANNALS OF MATHEMATICS, ser. 2, vol. 2 (1900/01), p. 97. A geometrical construction for this curve is given by Eagles, *Constructive Geometry of Plane Curves*, p. 333.

the locus of the actual brilliant points is represented by a heavy full line and that of the virtual brilliant points by a heavy dashed line. The spokes of the wheel being radial and of small radius, our family of curves is represented by the equation

$$y - ax = 0. \tag{10}$$

$$\therefore \qquad \frac{\partial F}{\partial x} = -a, \qquad \frac{\partial F}{\partial y} = 1.$$

Substituting these values in equation (b'), and eliminating a between the equation thus obtained and equation (10), we get

$$\Big[(x_2 y - y_2 x) + (y_1 x - x_1 y) + (x_1 y_2 - x_2 y_1)\Big](-1)\Big[\{x_1 y - y_1 x\}\{x(x - x_2)$$
$$+ y(y - y_2)\} + \{x_2 y - y_2 x\}\{x(x - x_1) + y(y - y_1)\}\Big] +$$
$$z_1^2\{x(x - x_2) + y(y - y_2)\}^2 - z_2^2\{x(x - x_1) + y(y - y_1)\}^2 = 0. \tag{11}$$

The result may be stated as follows : *The brilliant curve of a family of radiating right lines, when the source and the recipient are not in the plane of the family, is a curve of the fourth degree.* This curve passes through the *radiant*, and also through the points P_i and P_j, whose coordinates have been given in §3. The first term of equation (11) differs from that of equation (4) only in sign.

When $z_1 = 0$ and $z_2 = 0$, the curve represented by equation (11) degenerates into the curves represented by equations (8), (9). This is in accordance with the theorem of §4, since the families of curves represented by equations (3) and (10) are orthogonal trajectories.

(*c*) *A Family of Equilateral Hyperbolas.* Let it be required to find the brilliant curve of the family of plane curves

$$xy = a, \tag{12}$$

when the source and recipient are not in the plane of the curves.

$$\frac{\partial F}{\partial x} = y, \qquad \frac{\partial F}{\partial y} = x.$$

Substituting these values in (b'), we get

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_1 & 1 \end{vmatrix} \Big[\{y(x-x_1)+x(y-y_1)\}\{y(y-y_2)-x(x-x_2)\}+\{y(x-x_2)$$

$$+x\,(y-y_2)\}\{y(y-y_1)-x(x-x_1)\}\Big]+z_1^2\{y(y-y_2)-x(x-x_2)\}^2$$

$$-z_2^2\{y(y-y_1)-x(x-x_1)\}^2=0 \qquad (13)$$

6. Geometrical Constructions of the Brilliant Curve for Certain Families of Plane Curves. It is a property of the sphere that, if any point of its surface be connected by right lines with a pair of conjugate points and also with the points in which the sphere is pierced by the diameter containing the conjugate points, the latter lines bisect internally and externally the angle formed by the former.

We can make use of this property to test whether a point P of a space curve is a brilliant point with respect to a source P_1 and a recipient P_2. At P draw a plane normal to the curve and through the point P_a in which this plane is pierced by the right line $P_1 P_2$ draw a sphere having P_1 and P_2 as conjugate points. The point P is or is not a brilliant point according as this sphere does or does not pass through it. If the normal plane is parallel to the right line $P_1 P_2$, the sphere is replaced by the plane which is perpendicular to $P_1 P_2$ at the point half way between P_1 and P_2. If the normal plane contains both P_1 and P_2, the point P is by definition a brilliant point. On this principle is based a geometrical construction for finding the brilliant curve of a family of parallel curves, whose common plane we shall call the horizontal plane, with respect to a source and a recipient which are not in the horizontal plane:

To any curve, at any point, draw a normal; this line will be normal to every curve of the family. At the point in which it cuts the horizontal projection of $P_1 P_2$ draw a vertical line, and through the point in which this line cuts the right line $P_1 P_2$ draw a sphere having P_1 and P_2 as conjugate points. This sphere cuts the horizontal plane in a circle which cuts the normal in two points of the required locus.

In Fig. 2 this method is used for finding the brilliant curve of a family of concentric circles.

From the second definition of a brilliant point (see Condition 3) we

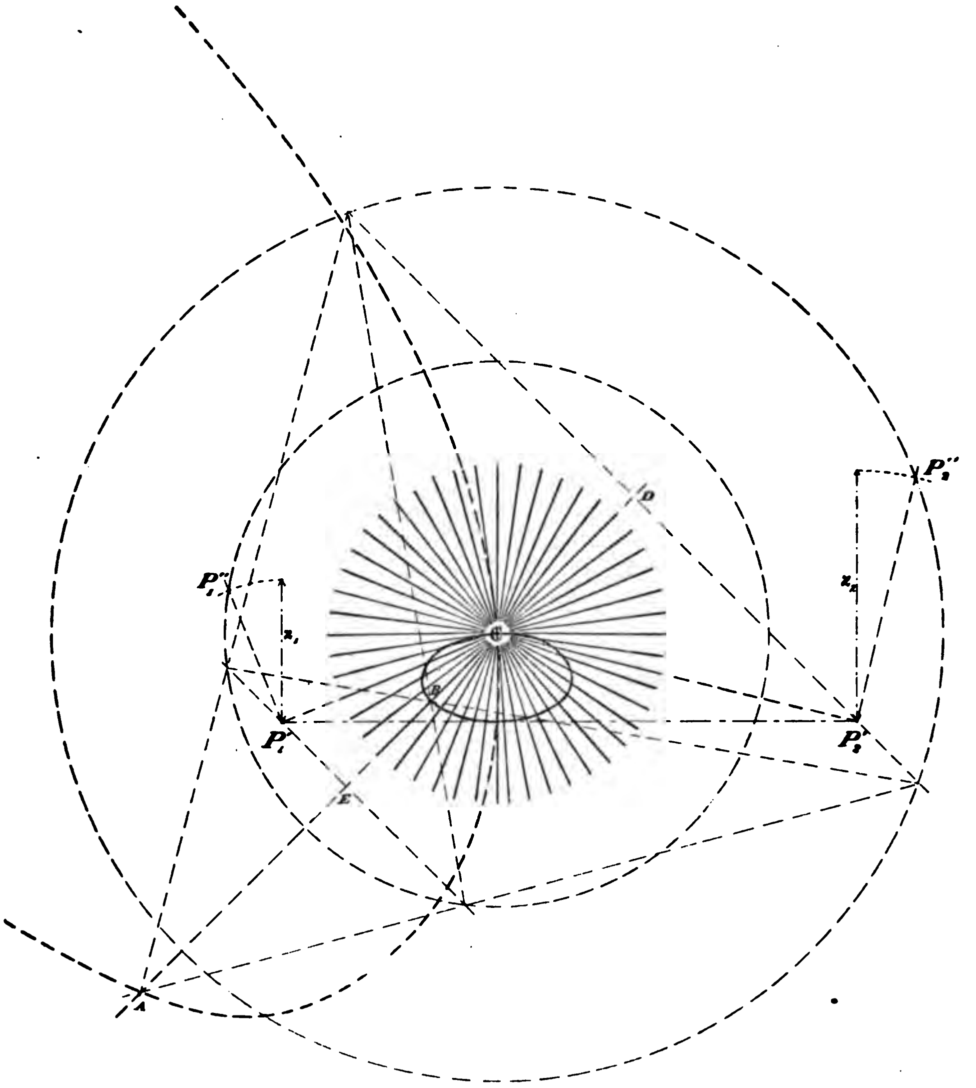

FIG. 3.

obtain the following geometrical construction for finding the brilliant points of a right line l with respect to a point source P_1 and a point recipient P_2.

About l as an axis revolve P_1 until it comes into the plane of l and P_2. There will be two revolved positions. Connect by right lines P_2 with each of the revolved positions of P_1. These right lines cut the line l in its brilliant points.

To find the locus of the brilliant points of a family of right lines, find the brilliant points of each member of the family separately.

In Fig. 3 this method is used for finding the brilliant curve of a family of radiating right lines, all of which lie in one plane, with respect to a source and recipient not in this plane.

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PROBLEMS IN INFINITE SERIES AND DEFINITE INTEGRALS; WITH A STATEMENT OF CERTAIN SUFFICIENT CONDITIONS WHICH ARE FUNDAMENTAL IN THE THEORY OF DEFINITE INTEGRALS.

By W. F. Osgood.

Simple sufficient conditions that a convergent series of continuous real functions of a real variable (or variables)

$$u_1(x) + u_2(x) + \cdots$$

may represent a continuous function and may be integrated or differentiated term by term are now pretty generally known ;—I refer to the conditions stated at the beginning of §1. That equally simple conditions, closely analogous to these, exist for the *Definite Integral*

$$\int_a^{\infty} f(x, \boldsymbol{a})\, dx$$

is less generally known and it is believed that a statement of these conditions will be welcome to the readers of the Annals.* Here $\boldsymbol{a}$ shall lie in the interval (a, b) and $f(x, \boldsymbol{a})$ shall be a continuous function of the two independent real variables $x, \boldsymbol{a}$ for all values of these variables belonging to the domain

$$S : \quad \begin{cases} a \leqq \boldsymbol{a} \leqq b, \\ g \leqq x. \end{cases}$$

It is assumed that the integral converges for every value of $\boldsymbol{a}$ in the interval (a, b) ; *i. e.* that, $\boldsymbol{a}$ being given any such value and then held fast, the integral

$$\int_g^x f(x, \boldsymbol{a})\, dx$$

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* These conditions were obtained by Ch.-J. de la Vallée-Poussin in a noted paper : Étude des intégrales à limites infinies pour lesquelles la fonction sous le signe est continue, *Annales de la Société scientifique de Bruxelles*, vol. 16 (1891/92), p. 150; continued in *Journal de mathématiques*, ser. 4, vol. 8 (1892), p. 421. *Cf.* also C. Jordan, *Cours d'analyse*, vol. 2, 2d ed., §§64–72; O. Stolz, *Diff.- u. Integralrechnung*, vol. 3, chap. 16.

It is to be regretted that the late Professor Brunel in his excellent article on Definite Integrals, *Encyklopädie der mathematischen Wissenschaften*, vol. 2, p. 135, did not state these conditions. He gives but one of them, and that, one of the more elaborate for dealing with the special integral of §3 below.

will approach a limit when x becomes infinite. This limit will be a function of α; we write it as $\phi(\alpha)$. By definition, then,

$$\phi(\alpha) = \lim_{x=\infty} \int_g^x f(x, \alpha)dx = \int_g^\infty f(x, \alpha)dx.$$

The questions which present themselves are:

(a) When will the function $\phi(\alpha)$ be continuous?

(b) If continuous, when can it be integrated by integrating the integral under the sign of integration, *i. e.* when will

$$\int_{\alpha_0}^{\alpha} \phi(\alpha)d\alpha = \int_g^\infty dx \int_{\alpha_0}^{\alpha} f(x, \alpha)dx \quad ?$$

(c) When can $\phi(\alpha)$ be differentiated by differentiating the integral under the sign of integration; *i. e.* when will

$$\frac{d\phi(\alpha)}{d\alpha} = \int_g^\infty \frac{\partial f(x, \alpha)}{\partial \alpha} dx \quad ?$$

In the great majority of cases that arise in practice the answer to these questions is not difficult. It is stated below in §2.

§1.

THE TESTS FOR AN INFINITE SERIES.

THEOREM A. *If each term of the series*

$$u_1(x) + u_2(x) + \cdots$$

is a continuous function of the real variable x in the interval $a \leqq x \leqq b$ and if the series converges uniformly in this interval, then the function $f(x)$ represented by the series is continuous in this interval.

THE WEIERSTRASSIAN M-TEST. *The above series will converge uniformly if there exists a convergent series of positive constants, $\sum_{n=1}^{\infty} M_n$, such that, no matter what value x may have in the interval,*

$$| u_n(x) | \leqq M_n, \qquad n = 1, 2, \cdots$$

THEOREM B. *If the above series converges uniformly, it can be integrated term by term:*

$$\int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{x_1} u_1(x)dx + \int_{x_0}^{x_1} u_2(x)dx + \cdots,$$

where x_0, x_1, are any two numbers of the interval (a, b).

Theorem C. *If each term of the convergent series*

$$f(x) = u_1(x) + u_2(x) + \cdots$$

has a derivative $u_n'(x)$ which is continuous throughout the interval $a \leqq x \leqq b$ and if the series of the derivatives

$$u_1'(x) + u_2'(x) + \cdots$$

converges uniformly throughout this interval, then $f(x)$ has a derivative at each point of the interval and the derivative is given by the formula:

$$f'(x) = u_1'(x) + u_2'(x) + \cdots .$$

Each of the above four conditions is sufficient; no one of them is necessary. Theorem B is not in general true if a or b is infinite and the integral is extended to infinity.

§2.

The Tests for a Definite Integral.

Let $f(x, \alpha)$ be a continuous function of the real variables x, α at all points of the region

$$S: \quad \begin{cases} a \leqq \alpha \leqq b, \\ g \leqq x. \end{cases}$$

The integral

$$\int_g^h f(x, \alpha)\, dx,$$

h being a constant, is a *proper integral* and it defines a continuous function of α, $\phi(\alpha)$. This appears at once if we interpret the integral geometrically as the area under the curve in which the surface $z = f(x, y)$ is cut by the plane $y = \alpha$.*

If, however, the upper limit of integration is infinite, we have to do with an *improper integral*, and it by no means follows, even if this integral:

$$\int_g^\infty f(x, \alpha)\, dx$$

converges for all values of α in question, that the function which it represents, $\phi(\alpha)$, is continuous. For example, we see by direct computation that the integral

$$\int_0^\infty \alpha e^{-\alpha x}\, dx$$

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* For an analytic proof *cf.* Picard, *Traité d'analyse*, vol. 1, p. 29.

converges for all values of $\alpha \geqq 0$; but the function which it represents, $\phi(x)$, is discontinuous:

$$\begin{aligned} \phi(\alpha) &= 1, & \alpha &> 0, \\ \phi(\alpha) &= 0, & \alpha &= 0. \end{aligned}$$

Again, the integral

$$\int_0^\infty \sin \pi\alpha \, e^{-\sin^2 \pi\alpha x} \, dx,$$

obtained from the foregoing in a simple manner, converges for all values of α; but it is discontinuous for every positive or negative integral value of α, inclusive of zero.

Under what conditions, then, will the integral

$$\int_g^\infty f(x, \alpha) dx,$$

assumed convergent, represent a continuous function of α? The answer to this and similar questions is given in the theorems that follow.

Definition of Uniform Convergence of an Improper Definite Integral. The integral

$$\int_g^\infty f(x, \alpha) dx,$$

where $f(x, \alpha)$ is continuous throughout the region S, is said to *converge uniformly* throughout the interval $a \leqq \alpha \leqq b$ if, a positive quantity ϵ being chosen at pleasure, there exists a positive quantity G, independent of α, such that

$$\left| \int_G^x f(x, \alpha) dx \right| < \epsilon, \qquad x > G.$$

THEOREM A. *If $f(x, \alpha)$ is a function of the independent real variables x, α continuous throughout the region S, and if the integral*

$$\int_g^\infty f(x, \alpha) dx$$

converges uniformly throughout the interval $a \leqq \alpha \leqq b$, then the function $\phi(\alpha)$ represented by the integral is continuous throughout this interval.

THE VALLÉE-POUSSIN $\mu(x)$-TEST. *The above integral will converge uniformly if there exists a function* $\mu(x)$, *independent of* α, *which satisfies the following conditions:*

$$\begin{aligned}
&(a) \quad \mu(x) \geqq 0, \quad x \geqq g;\\
&(b) \quad |f(x,\alpha)| \leqq \mu(x), \quad x \geqq g;\\
&(c) \quad \int_g^{\infty} \mu(x)\,dx \quad \textit{converges.}
\end{aligned}$$

COROLLARY. *These conditions will always be satisfied if there exists a constant* $k > 1$ *such that the product*

$$x^k f(x,\alpha)$$

remains finite throughout the region S.

THEOREM B. *If the above integral converges uniformly, it can be integrated under the sign of integration:*

$$\int_{\alpha_0}^{\alpha_1}\phi(\alpha)d\alpha = \int_g^{\infty}dx\int_{\alpha_0}^{\alpha_1}f(x,\alpha)d\alpha,$$

where α_0, α_1, *are any two numbers of the interval* (a, b).

THEOREM C. *If the above function has a first partial derivative with regard to* α, $f_\alpha(x,\alpha)$, *at each point of* S, *continuous throughout* S, *and if*

$$\int_g^{\infty} f_\alpha(x,\alpha)dx$$

converges uniformly throughout the interval $a \leqq \alpha \leqq b$; *then the function* $\phi(\alpha)$ *has a derivative at each point of the interval, and*

$$\frac{d\phi(\alpha)}{d\alpha} = \int_g^{\infty} f_\alpha(x,\alpha)dx.$$

Each of the above four conditions is sufficient; no one of them is necessary. Theorem B is not in general true if a or b is infinite and the integral with regard to α is extended to infinity.

Other Improper Integrals. In the foregoing, improper integrals were considered, one of whose limits of integration is infinite, but whose integrand is continuous throughout S. Similar theorems hold for integrals taken between finite limits, the integrand becoming infinite at one of the limits of integration.

Thus, let $f(x, \alpha)$ be a continuous function of x, α at all points of the region

$$R: \quad \begin{cases} a \leqq \alpha \leqq b, \\ g \leqq x < h: \end{cases}$$

but suppose that $f(x, \alpha)$ becomes infinite in the points $x = h$, $\alpha = \alpha$, the integral now being

$$\int_g^h f(x, \alpha) dx.$$

Then the definition of uniform convergence will read, *mutatis mutandis*, precisely as before except that G is now taken near to h and x lies between G and h. The four tests hold as before, with obvious modifications.

The proofs of the foregoing theorems are similar to the proofs of the corresponding theorems of § 1 and serve as a useful exercise for the student who is familiar with these proofs and is just beginning the study of definite integrals from a modern standpoint.

§3.

A SPECIAL SERIES AND A SPECIAL INTEGRAL.

I. *The Series.* Let each term of the series

$$u_1(x) + u_2(x) + \cdots$$

be continuous for all values of $x \geqq g$ and let the series converge uniformly throughout any arbitrarily assigned interval $g \leqq x \leqq h$. Then, if $f(x)$ denotes the function represented by the series, $f(x)$ is continuous and

$$\int_g^x f(x) dx = \int_g^x u_1(x) dx + \int_g^x u_2(x) dx + \cdots.$$

When can this series be integrated term by term to infinity? We must, of course, first of all require that each term be integrable to infinity: *i. e.*

1° $\displaystyle\int_g^\infty u_n(x) dx$ shall converge, $\qquad n = 1, 2, \ldots$

Furthermore let the above series of integrals converge uniformly throughout the *unlimited* interval $x \geqq g$: *i. e.*

2° A positive quantity ϵ being chosen arbitrarily small, it shall then be possible to find a positive integer independent of x such that for all values of $x \geqq g$.

$$\left| \int_g^x u_{m+1}(x)dx + \cdots + \int_g^x u_{m+p}(x)dx \right| < \epsilon, \qquad p = 1, 2 \cdots$$

Then it is easy to show that the series can be integrated to infinity term by term; or, more fully,

(α) that the series of limits

$$\int_g^\infty u_1(x)dx + \int_g^\infty u_2(x)dx + \cdots$$

converges;

(β) that the integral $\int_g^\infty f(x)\, dx$ converges;

(γ) that

$$\int_g^\infty f(x)dx = \int_g^\infty u_1(x)dx + \int_g^\infty u_2(x)dx + \cdots$$

II. *The Integral.* A closely analogous sufficient condition holds for the reversal of the order of integration in the integral

$$\int_a^\infty d\alpha \int_g^\infty f(x, \alpha)\, dx,$$

where $f(x, \alpha)$ is a continuous function at all points (x, α) of the region

$$\bar{S}: \quad a \leqq \alpha, \quad g \leqq x.$$

1° *The integrals*

$$\int_g^\infty f(x, \alpha)dx, \qquad \int_a^\infty f(x, \alpha)d\alpha$$

shall respectively converge uniformly throughout every FIXED *interval*;

$$a \leqq \alpha \leqq b, \qquad g \leqq x \leqq h;$$

2° *The integral*

$$\int_a^\alpha d\alpha \int_g^\infty f(x, \alpha)dx$$

shall converge uniformly throughout the UNLIMITED *interval* $\alpha \geqq a$; *i. e. a positive quantity* ϵ *being chosen arbitrarily small, it shall then be possible to find a positive quantity* G, *independent of* α, *such that for all values of* $\alpha \geqq a$

$$\left| \int_a^\alpha d\alpha \int_G^x f(x, \alpha)\,dx \right| < \epsilon, \qquad x > G.$$

When these conditions are fulfilled

$$\int_a^\infty d\alpha \int_g^\infty f(x, \alpha)\,dx = \int_g^\infty dx \int_a^\infty f(x, \alpha)\,d\alpha.$$

It sometimes happens that one of the integrals of Condition 1°,—the first, let us say,—does not converge uniformly in the interval (a, b), but that it converges uniformly in every interval lying inside of the region $\alpha \geqq a$, namely, $a < a' \leqq \alpha \leqq b$, where a', b are arbitrary.

In this case, if the remainder of Condition 1° and Condition 2° are satisfied, the reversal of the order of integration is still allowable.

A more elaborate test for a case of importance in practice is also given by Vallée-Poussin and is stated in the *Encyklopädie*, *l. c.*

§4.

PROBLEMS IN INFINITE SERIES AND PRODUCTS.

A.—CONVERGENCE.

1. Is the infinite series:

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 + \cdots$$

convergent or divergent?

2. Obtain by means of Cauchy's integral test for the convergence of an infinite series of positive terms (*cf.* Picard, *Traité d'analyse*, vol. 1, p. 27) an upper limit for the value of the remainder of the series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots, \qquad p > 1.$$

Hence compute the value of this series when $p = 1\frac{1}{2}$ correct to two places of decimals. How many terms of this series would it be necessary to sum in order to obtain the same degree of approximation?

3. For what values of x does the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt[n]{n+n^n}}$$

converge? Discuss all cases.

4. Determine all values of x for which the series

$$\left(\frac{1}{3}\right)^2 x - \left(\frac{1\cdot 2}{3\cdot 5}\right)^2 x^2 + \left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2 x^3 \cdots$$

converges.

5. Is the series

$$\sum_{n=1}^{\infty} \frac{1}{\log(1+n)\cdot\log(1+n^n)}$$

convergent or divergent?

6. Show that if

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

represents a power series convergent for values of x other than $x = 0$, the series

$$a_0 + a_1 x + \frac{1}{2!} a_2 x^2 + \quad \cdots + \frac{1}{i!} a_i x^i + \cdots$$

will converge for all values of x.

For what values of x will the series

$$\sum_{i=1}^{\infty} \frac{1}{i^2} a_i x^i$$

converge?

B.—Series of Functions.

7. Let the series

$$u_1(x) + u_2(x) + \cdots\cdot$$

be a series of functions each continuous in the interval $a \leqq x \leqq b$ and let it converge uniformly in this interval. Show that the term-by-term integral of this series:

$$\int_a^x u_1(x)\,dx + \int_a^x u_2(x)\,dx + \cdots\cdot$$

is a uniformly convergent series throughout this interval.

8. Let $s_n(m)$ be a single valued function of the two positive integers m, n, which satisfies the following conditions:

(*a*) $\lim\limits_{n=\infty} s_n(m)$ exists; denote it by $f(m)$;

(*b*) $\lim\limits_{m=\infty} s_n(m)$ exists; denote it by S_n;

(*c*) $s_n(m)$ converges uniformly when $n = \infty$:

$$| s_n(m) - s_{n'}(m) | < \epsilon, \qquad n > \nu,\ n' > \nu.$$

Then

(1) $\lim\limits_{m=\infty} f(m)$ exists; denote it by A:

(2) $\lim\limits_{n=\infty} S_n$ exists; denote it by B;

(3) $A = B$.

9. Apply the theorem of Problem 8 to prove that

$$\lim_{n=\infty}\left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

10. From the formula

$$\cos mx = \cos^m x - \frac{m(m-1)}{1 \cdot 2}\cos^{m-2}x \sin^2 x + \cdots \cdot$$

deduce the development

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \cdot$$

and justify each step of your work.

11. The series

$$\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \cdots$$

converges when $x > 1$. Does the series represent a continuous function? Can it be differentiated term by term? Denoting the function represented by the series by $f(x)$, discuss rigorously the shape of the curve $y = f(x)$.

12. Let

$$a_0 + a_1 + \cdots$$

be an absolutely convergent series of constant terms, and let

$$f_0(x),\ f_1(x),\ \cdots$$

be a set of functions each continuous within the interval $\alpha \leqq x \leqq \beta$ and each comprised between certain fixed limits

$$A \leqq f_i(x) \leqq B, \qquad i = 0, 1, \ldots$$

where A, B are constants. Show that the series

$$a_0 f_0(x) + a_1 f_1(x) + \cdots$$

represents a continuous function of x within this interval: $\alpha \leqq x \leqq \beta$.

13. Let $s_n(x)$ be a continuous function of x in the interval $a \leqq x \leqq b$ and let $s_n(x)$ converge, for every rational value of x pertaining to this interval, toward a limit when n increases indefinitely. Furthermore, let

$$| s_n(x') - s_n(x) | < M | x' - x | ,$$

where M is a fixed positive quantity and x, x' are any two points of the interval. Prove that $s_n(x)$ converges uniformly toward a limit for all values of x in the interval.

14. If $$a_0 + a_1 + \cdots$$
and $$b_0 + b_1 + \cdots$$
are any two absolutely convergent series, show that the value of the series

$$a_0 b_0 + a_1 b_1 + \cdots$$

is given by the integral:

$$\frac{1}{2\pi} \int_0^{2\pi} f_1(e^{\theta i})\, f_2(e^{-\theta i})\, d\theta,$$

where

$$f_1(x) = \sum_0^\infty a_i x^i, \qquad f_2(x) = \sum_0^\infty b_i x^i.$$

15. If the terms of the series

$$u_1(x) + u_2(x) + \cdots$$

are all uniformly continuous throughout the interval $a < x < b$ and if the series converges uniformly throughout this interval, show that the function $f(x)$ represented by the series approaches a limit when x approaches a.

16. Given that the series

$$u_1(x) + u_2(x) + \cdots$$

converges uniformly throughout the interval $a < x < b$ and that it converges when $x = a$. Can we infer that it converges uniformly when $a \leqq x < b$?

17. If the terms of the series

$$u_1(x) + u_2(x) + \cdots$$

are all continuous throughout the interval $a \leqq x \leqq b$ and if the series converges uniformly throughout the interval $a < x \leqq b$, will it necessarily converge when $x = a$, and, if it does converge, will the function that it represents necessarily be continuous at the point $x = a$?

18. If the terms of the series

$$u_1(x) + u_2(x) + \cdots$$

are all continuous throughout the interval $a \leqq x \leqq b$ and if the series converges uniformly throughout every interval $\alpha \leqq x \leqq \beta$ lying inside of the interval (a, b): $a < \alpha < \beta < b$, will the series necessarily converge when $x = a$, and, if it does converge, will the function that it represents necessarily be continuous at the point $x = a$?

19. Show that the series

$$\frac{1}{\sqrt{1 - 2r\mu + r^2}} = \sum_0^\infty r^n P_n(\mu), \qquad \begin{cases} -1 \leqq \mu \leqq 1, \\ -1 < r < 1, \end{cases}$$

can be differentiated term by term both as regards r and as regards μ.

Suggestion: Prove the theorem first for the value of μ in question and for a value of r numerically small; and then show that the theorem still holds when r has the value in question.

20. Given the formula

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)} \left\{ 1 - \frac{x^2}{2\cdot(2n+2)} + \frac{x^4}{2\cdot4\cdot(2n+2)(2n+4)} - \cdots \right\}$$

show that

$$\frac{\partial^2 J_n(x)}{\partial n\, \partial x} = \frac{\partial^2 J_n(x)}{\partial x\, \partial n}.$$

21. Show that if the series

$$V = \phi_0(x, y, z) + \phi_1(x, y, z) + \phi_2(x, y, z) + \cdots\cdots,$$

where $\phi_n(x, y, z)$ denotes a homogeneous polynomial in (x, y, z) of degree n, represents a function satisfying Laplace's equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

then each function $\phi_n(x, y, z)$ must also satisfy Laplace's equation.

Suggestion: Assume that such a function V can be developed into a triple power series in x, y, z for values of the arguments numerically sufficiently restricted, the development holding for all values of the arguments in the neighborhood of the point $(0, 0, 0)$.

22. Show that if

$$u_0(x) + u_1(x) + \cdots$$

is a series of functions each continuous and having no roots in the interval $a \leqq x \leqq b$, and if

$$\left| \frac{u_{n+1}(x)}{u_n(x)} \right| \leqq \gamma < 1, \qquad n \geqq m,$$

where γ, m, do not depend on x, then the given series is uniformly convergent within the interval.

Apply this test to the following series:

$$1 + xa + \frac{x(x-1)}{2!}a^2 + \frac{x(x-1)(x-2)}{3!}a^3 + \cdots$$

where $\qquad 0 < a < 1.$

23. Find all the values of x for which the series

$$e^x \sin x + e^{2x} \sin 2x + e^{3x} \sin 3x + \cdots$$

converges. Does it converge uniformly for these values?

For what values of x can the series be differentiated term by term?

24. For what values of x is the series

$$1 + x + x^2 + \cdots$$

absolutely convergent? Is it uniformly convergent throughout the same interval?

25. Does the series

$$x + \frac{x^3}{3} - \frac{x^2}{2} + \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^4}{4} + \frac{x^9}{9} + \frac{x^{11}}{11} - \frac{x^6}{6} + \cdots$$

converge uniformly in the interval $0 \leqq x \leqq 1$?

26. Show that the series

$$\frac{1}{(1+x)^2} + \frac{1}{(2+x)^2} + \frac{1}{(3+x)^2} + \cdots, \qquad 0 \leqq x,$$

can be integrated term by term between any two positive finite limits.

Can the series be integrated term by term between the limits 0 and ∞? Can the function defined by the series be integrated between these limits?

27. Show that the series

$$2xe^{-x^2} = \sum_{n=1}^{\infty}\left(\frac{1}{n^2}e^{-\frac{x^2}{n^2}} - \frac{1}{(n+1)^2}e^{-\frac{x^2}{(n+1)^2}}\right)$$

can be integrated term by term between any two finite limits.

Can the function defined by the series be integrated between the limits 0 and ∞? If so, is the value of this integral given by integrating the series term by term between these limits?

28. If each term of the series

$$u_1(x) + u_2(x) + \cdots$$

is a continuous function of x for all values of $x \geqq a$, and if the series

$$x^k u_1(x) + x^k u_2(x) + \cdots, \qquad k > 1,$$

satisfies the M-test (§ 1), then the given series may be integrated term by term from a to ∞.

29. Show that the series

$$e^{-x^2} - \frac{x^2e^{-x^2}}{2!} + \frac{x^4e^{-x^2}}{4!} - \frac{x^6e^{-x^2}}{6!} + \cdots$$

can be integrated term by term from 0 to ∞, given that $(n-1)! = n^{n-\frac{1}{2}}e^{-n}K_n$, when K_n lies between two positive constant quantities.

30. Can the function defined by the series

$$\frac{1}{(1+x)^3} + \frac{1}{(2+x)^3} + \frac{1}{(3+x)^3} + \cdots$$

be integrated from 0 to ∞, and if so, is its value given by the term-by-term integral of the series?

C.—Double Series.

31. Show that the function

$$\log(1 + e^x)$$

can be developed into a power series in x. State carefully each theorem that you employ.* Determine the coefficients of the first four terms in the development.

32. Develop the function

$$f(x) = \int_0^{\frac{\pi}{2}} \sin^{-1}(x \sin\phi)\, d\phi$$

into a power series in x. Justify each step. Assume the formula

$$\int_0^{\frac{\pi}{2}} \sin^n\phi\, d\phi = \frac{2 \cdot 4 \cdot 6 \cdot \,\ldots\, \cdot (n-1)}{1 \cdot 3 \cdot 5 \cdot \,\ldots\, \cdot n},$$

where n is odd.

33. Develop $\sqrt[3]{\cos x}$ into a series proceeding according to ascending integral powers of x, stating accurately the theorems of which you make use. For how large a region will the series surely converge and represent the function?

Further problems in infinite series will be found in the exercises at the end of chs. 17 and 18 of Gibson's *Elementary Treatise on the Calculus.*

D. — Infinite Products.

34. Discuss the infinite product

$$(1 - \sin x)\,(1 + \sin \tfrac{1}{2}x)\,(1 - \sin \tfrac{1}{3}x) \;.\;.\;.\;.$$

with reference to absolute and to uniform convergence.† Study the function defined by this product and plot the corresponding curve.

---

* The final theorem here needed is to be found in Stolz, *Allgemeine Arithmetik*, vol. 1, ch. 10, §25. A special case of the theorem sufficiently general for the great majority of the cases that arise in practice (in particular, for the present case) is stated in my *Infinite Series*, §36.

Numerous examples of developments that can be obtained by this theorem are given in the text-books on Calculus; *cf.*, for example, Byerly, *Problems in Differential Calculus*, pp. 46–48. The analysis at the disposal of the student at that stage of his work is inadequate for complete demonstrations, and these are properly deferred for an advanced course in analysis.

† An infinite product $f_1 f_2 \ldots.$ is said to converge when and only when (*a*) the factors from a definite one, f_m, on are all different from zero; (*b*) the product $f_{m+1} f_{m+2} \cdots f_{m+p}$ approaches a limit different from zero.

An infinite product $f_1(x) f_2(x) \ldots$ is said to converge uniformly throughout the interval $a \leqq x \leqq b$ if, a positive quantity ϵ being chosen at pleasure, there exists a positive integer m independent of x and such that

$$| f_{m+1}(x)\, f_{m+2}(x) \,\ldots\, f_{m+p}(x) - 1 | < \epsilon, \qquad p = 1, 2, \ldots\ldots$$

These definitions can, of course, be replaced by more general ones. But the products that would thus be admitted it is not useful to consider in practice.

35. Show that the infinite product

$$\cos x \cdot \cos \tfrac{1}{2} x \cdot \cos \tfrac{1}{3} x \cdot \ . \ . \ .$$

converges for all values of x and discuss the function that it defines.

36. Discuss the infinite product

$$\left(1 - \frac{x}{1}\right)\left(1 + \frac{x}{3}\right)\left(1 - \frac{x}{5}\right)\left(1 + \frac{x}{7}\right) \cdot \cdot \cdot \cdot$$

with regard to absolute and to uniform convergence. Plot the function defined by the product as a curve. Find its slope when $x = 0$ and when $x = 1$.

37. Show that the above products can be developed into power series in x and determine the coefficients of the terms of degree 0, 1, 2.

§5.

Problems in Definite Integrals.

38. Show that the integral

$$\int_1^{\infty} x^{a-1} e^{-x} \, dx$$

converges and defines a continuous function for all values of a.

39. Show that the integral

$$\int_0^1 x^{a-1} e^{-x} \, dx$$

converges and defines a continuous function for all positive values of a.

Give the proof first by introducing as a new variable of integration $y = x^{-1}$; then by applying the theorems of §2 directly, modified for a finite domain of integration.

40. Show that the function

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} \, dx$$

is continuous for all positive values of a and that

$$\Gamma'(a) = \int_0^{\infty} x^{a-1} \log x \, e^{-x} dx$$

$$\Gamma''(a) = \int_0^{\infty} x^{a-1} (\log x)^2 e^{-x} \, dx$$

etc.

Plot the curve $y = \Gamma(\alpha)$, showing that it has a minimum between $\alpha = 1$ and $\alpha = 2$, and that it is always convex toward the α-axis.

41. The integral

$$\int_0^\infty \alpha^3 e^{-\alpha^2 x} dx$$

converges for all values of α and represents the function $\phi(\alpha) = \alpha$. Can the integral be differentiated under the sign of integration?

42. Given the formula

$$\int_0^\infty e^{-x^2} dx = \tfrac{1}{2}\sqrt{\pi},$$

deduce the formulas

$$\int_0^\infty e^{-\alpha x^2} dx = \tfrac{1}{2}\sqrt{\frac{\pi}{\alpha}},$$

$$\int_0^\infty x^2 e^{-\alpha x^2} dx = \tfrac{1}{4}\frac{\sqrt{\pi}}{\alpha^{\frac{3}{2}}}, \text{ etc.}$$

43. Show that

$$\int_0^\infty e^{-x^2} \cos 2\alpha x \, dx = \tfrac{1}{2}\sqrt{\pi}\, e^{-x^2}.$$

Suggestion: Denote the value of the function by u and show that

$$\frac{du}{d\alpha} + 2\alpha u = 0, \qquad u\big|_{\alpha=0} = \tfrac{1}{2}\sqrt{\pi}.$$

44. Show that

$$\int_0^\infty e^{-\left(x^2 + \frac{\alpha^2}{x^2}\right)} dx = \tfrac{1}{2}\pi e^{-2\alpha}.$$

Is it allowable here to differentiate under the sign of integration for all values of α? If not, and if you bave used the method suggested in Problem 43, have you been careful to make your solution rigorous?

45. Show 1) that the integral

$$\int_0^\infty \frac{\cos \alpha x}{1 + x^2} dx$$

defines a continuous function, $f(\alpha)$, for all values of α; 2) that

$$\phi(\alpha) = \int_0^\alpha f(\alpha)\, d\alpha$$

is given by integration under the integral sign with respect to α; 3) that $f'(\alpha)$ is given in general by differentiating under the integral sign with respect to α; 4) that $\phi(\alpha)$ satisfies the differential equation:

$$\frac{d^2\phi}{d\alpha^2} = \phi - \frac{\pi}{2} \qquad \text{when } \alpha > 0.$$

Hence determine the function $f(\alpha)$ defined by the integral.

46. Discuss the function defined by the integral:

$$\int_0^\infty e^{-\alpha^2 x^2} \sin x \, dx.$$

47. Discuss the function

$$\phi(\alpha) = \int_0^{\frac{\pi}{2}} \tan^\alpha \phi \, d\phi,$$

plotting the graph and determining the essential characteristics.

48. Show that the functions

$$f(\alpha) = \int_0^\infty e^{-x^2} \cos(\alpha x^2) dx,$$

$$\phi(\alpha) = \int_0^\infty e^{-x^2} \sin(\alpha x^2) dx,$$

satisfy the linear differential equations:

$$f'(\alpha) = \frac{-1}{2(1+\alpha^2)} \Big[\alpha f(\alpha) + \phi(\alpha)\Big],$$

$$\phi'(\alpha) = \frac{1}{2(1+\alpha^2)} \Big[f(\alpha) - \alpha\phi(\alpha)\Big].$$

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NOTE ON THE PRODUCT OF LINEAR SUBSTITUTIONS.*

By H. B. Newson.

If two linear substitutions in n variables be compounded, the product is also a linear substitution in n variables. The following method of expressing the result in determinant form is believed to be new. The proof is given for two substitutions in three variables, but the method and result are capable of immediate generalization for n variables.

Let T and T_1 be two substitutions as follows:

$$T:\ \begin{aligned} \rho x_1 &= a_1 x + b_1 y + c_1 z, \\ \rho y_1 &= a_2 x + b_2 y + c_2 z, \\ \rho z_1 &= a_3 x + b_3 y + c_3 z, \end{aligned} \qquad T_1:\ \begin{aligned} \rho_1 x_2 &= \alpha_1 x_1 + \beta_1 y_1 + \gamma_1 z_1, \\ \rho_1 y_2 &= \alpha_2 x_1 + \beta_2 y_1 + \gamma_2 z_1, \\ \rho_1 z_2 &= \alpha_3 x_1 + \beta_3 y_1 + \gamma_3 z_1. \end{aligned}$$

The substitution T_2 is obtained by eliminating x_1, y_1, z_1 from the above equations. This may be done as follows: Find the inverse of T by solving the three equations of T for x, y, z, Thus we get

$$T^{-1}:\ \begin{aligned} \frac{\Delta}{\rho} x &= A_1 x_1 + A_2 y_1 + A_3 z_1, \\ \frac{\Delta}{\rho} y &= B_1 x_1 + B_2 y_1 + B_3 z_1, \\ \frac{\Delta}{\rho} z &= C_1 x_1 + C_2 y_1 + C_3 z_1, \end{aligned}$$

where Δ is the determinant of T and A, B, etc. have the usual meanings.

The three equations of T^{-1} and the first one of T_1 form a system of four simultaneous linear equations; hence

$$\begin{vmatrix} -\frac{\Delta}{\rho} x & A_1 & A_2 & A_3 \\ -\frac{\Delta}{\rho} y & B_1 & B_2 & B_3 \\ -\frac{\Delta}{\rho} z & C_1 & C_2 & C_3 \\ -\rho_1 x_2 & \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix} = 0.$$

---

* Read before the Chicago Section of the American Mathematical Society at the Evanston meeting, 2/3 January, 1902.

This equation expresses the relation between x, y, z and x_2. Solving the last equation for x_2 we get

$$\rho\rho_1\Delta x_2 = \begin{vmatrix} x & A_1 & A_2 & A_3 \\ y & B_1 & B_2 & B_3 \\ z & C_1 & C_2 & C_3 \\ 0 & \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}.$$

In like manner we get similar results for y_2 and z_2; thus

$$\rho\rho_1\Delta y_2 = \begin{vmatrix} x & A_1 & A_2 & A_3 \\ y & B_1 & B_2 & B_3 \\ z & C_1 & C_2 & C_3 \\ 0 & \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix}; \qquad \rho\rho_1\Delta z_2 = \begin{vmatrix} x & A_1 & A_2 & A_3 \\ y & B_1 & B_2 & B_3 \\ z & C_1 & C_2 & C_3 \\ 0 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

When these three determinants are expanded, Δ divides out of both sides of the equation.

The general formula for n variables is

$$\rho\rho_1\Delta^{n-2} x_i''' = \begin{vmatrix} x_1 & A_1 & A_2 \, . \, . \, . \, . & A_n \\ x_2 & B_1 & B_2 \, . \, . \, . \, . & B_n \\ . & . & . & . \\ . & . & . & . \\ x_n & N_1 & N_2 \, . \, . \, . \, . & N_n \\ 0 & \alpha_i & \beta_i \, . \, . \, . \, . & \nu_i \end{vmatrix} \quad (i = 1 \cdot \cdot \cdot n).$$

THEOREM. *The value of x_i'' in the product of T and T_1, two linear substitutions, is proportional to the determinant formed by bordering the determinant of T^{-1}, the inverse of T, vertically by the variables of T and horizontally by the coefficients of the ith equation in T_1.*

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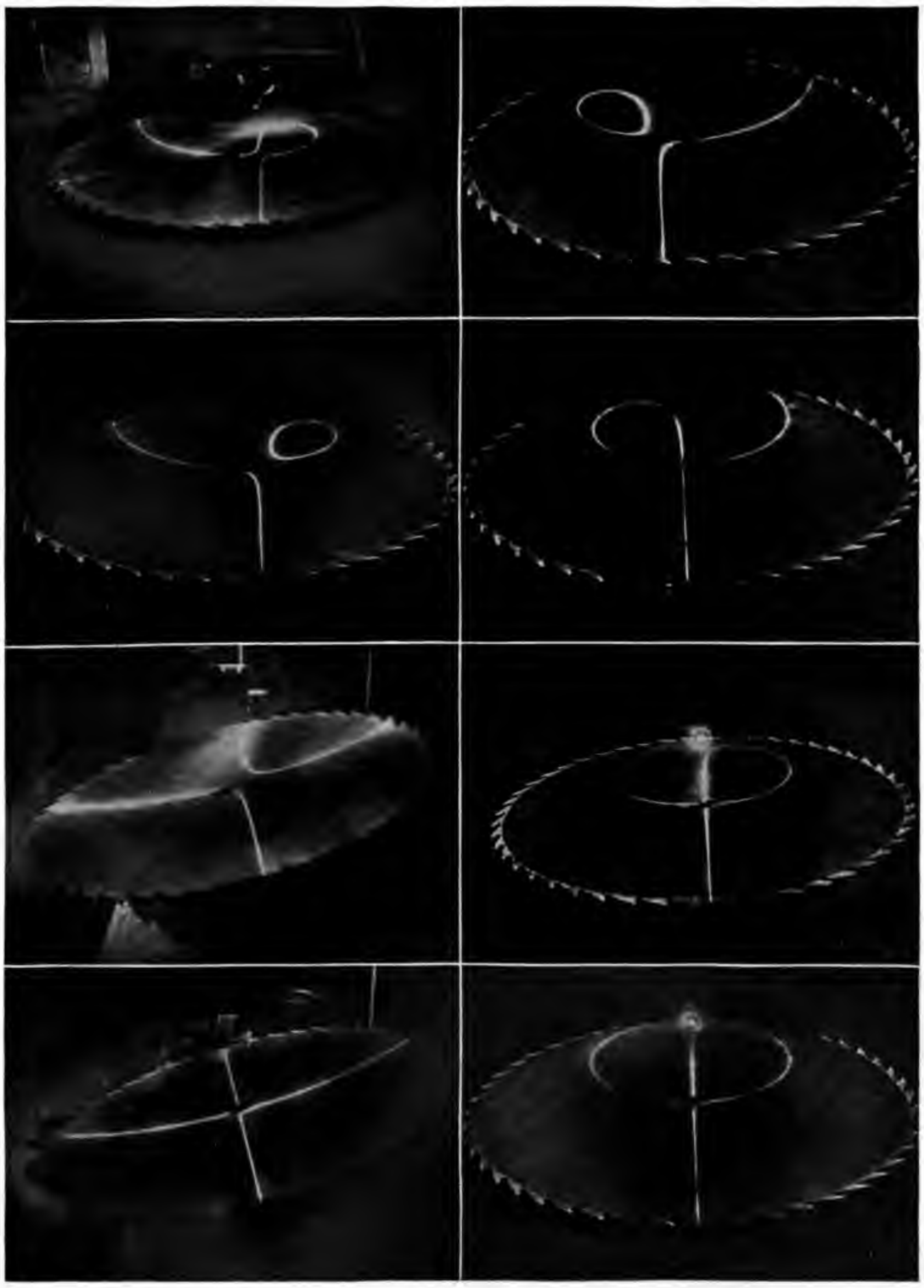
LAWRENCE, KANSAS.



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NOTE ON A TWISTED CURVE CONNECTED WITH AN INVOLUTION OF PAIRS OF POINTS IN A PLANE.[*]

BY H. S. WHITE.

1. A Method for Generating some Twisted Curves. The points of a plane are said to be in involution when they are so related in sets of n points that any one point of a set determines all the other $n-1$ points of that set. The simplest involutions, and the best known, are those where each set contains only 2 points. Of involutions of higher order so much is known as that they are all rational; that is, that between the points of a plane and the sets of points of any involution a one-to-one relation can be established.[†] But for orders higher than 2 the classification and reduction to types have not yet been given. Hence we shall understand for the present by "involution" always an involution of order 2, one whose points are grouped in pairs.

Bertini showed that involutions of second order are of four distinct types, every such involution falling into some one of these four classes.[‡] Two of the same type are usually not projectively equivalent, but each can be transformed into the other by some Cremona transformation. As the representative of each type he chooses the simplest. The first type is that in which collineation or projection effects an exchange of the points of each pair. Such a collineation evidently must be a perspectivity, or central projection. The second type includes as many distinct species as there are possible orders of curves, one for every integer above 3. Among these we consider that of lowest order, in which paired points lie on rays passing through a fixed point, and are conjugate with respect to a fixed conic.[§] These two alone are to be used in the present note.

The notion of an involution in a plane may be used to generate twisted curves, just as the notion of involution on a line has been used (and may be used further) to generate plane curves. To make this explicit, recall the method proposed by Steiner, and elaborated by Schroeter, for generating a cubic curve in the plane. A quadric involution on a straight line, it will be remembered, consists of pairs of points harmonic with respect to two fixed points, either real or imaginary, on the line. Two pairs of points determine fully such an involution, since there is only a single pair of points separating harmonically two given pairs. Any third pair of points, in order to belong to the same involution, must satisfy a single condition. Hence if three pairs of points are taken at random in a plane, and these are projected from a variable center in the plane upon any line, the requirement that their projections shall form 3 pairs in quadric involution will subject the variable center to a single condition; *i. e.* the locus of that point is a definite curve. In this case the locus is a general cubic curve.

---

[*] Read before the Chicago Section of the American Mathematical Society at its meeting January 3, 1902.

[†] G. Castelnuovo, *Mathematische Annalen*, vol. 44 (1894), p. 125.

[‡] Ricerche sulle trasformazioni univoche involutorie nel piano. *Annali di matematica*, ser. 2, vol. 8 (1877), p. 254.

[§] Compare the system of intersections of the circles of a net, used as an illustration, by Professor Bôcher in the *Annals of Mathematics*, ser. 2, vol. 3 (1902), pp. 49–52, and especially the first footnote on p. 52.

In the same way, if a certain number of pairs of points in a plane suffice to determine an involution of a specified type, then one additional pair of points, in order to belong in the same involution, must satisfy two conditions. Assume now, in space of 3 dimensions, pairs of points one more than sufficient; project them upon a plane from a variable center, and require the projections to form pairs in an involution of the specified type. This will be equivalent to 2 conditions, restricting the variable center to motion along some definite twisted curve.

In this way involutions of order 2 can give rise to an infinite variety of algebraic twisted curves, and may be found to lead to important properties of those curves. Indeed two species of involution in the plane which are equivalent to each other by virtue of some Cremona transformation will usually be connected with twisted curves which are not transformable into one another by any Cremona transformation of three dimensional space.

To exhibit the method I have worked out the following two simplest examples.

2. First Example: the Perspective Involution. In a perspective relation of the points in a plane, the axis o and the center O may be any line and any point. To find the conjugate to any point P_1 (see Fig. 1), produce a line OP_1 to cut the axis o in a point P', and determine a fourth point P_2 on OP_1 harmonic to O, P_1, and P'. All the pairs of points P_1P_2 constitute then an involution of the first type. Otherwise, two such pairs may be taken arbitrarily, and from them the center, the axis, and consequently all other pairs can be found. Suppose A_1 and A_2, B_1 and B_2 to denote given pairs, then any third pair $P_1 P_2$ must have their joining line pass through the point O where the line $A_1 A_2$ meets $B_1 B_2$. This is the first condition. Again, this center O determines two points A' and B' where the axis o must intersect $A_1 A_2$ and $B_1 B_2$; the second condition is that the center O and axis o shall separate harmonically the points P_1 and P_2.

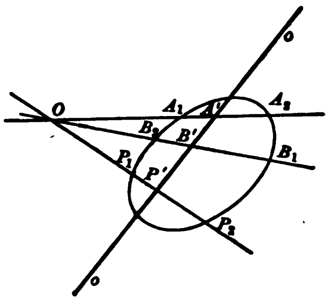

FIG. 1.

Assume in space three pairs of points $A_1 A_2$, $B_1 B_2$, $C_1 C_2$ (no four in any plane). If the projections of the 3 lines a, b, c which join these several pairs are to meet in one point of a plane, the center of projection X must lie on a line intersecting all three: a, b, and c. That is, X must lie on a ruled quadric having a, b, c as directrices. This first condition restricts X to a quadric surface.

The second condition is satisfied if the six projecting rays XA_1, XA_2, XB_1, . . . XC_2, lie on a quadric cone. For then the generator which passes through X, and its polar plane with respect to the cone, will project from X into the required center and axis of perspectivity, inasmuch as their sections by any plane will separate harmonically the projections of each pair of given points, *e. g.* of A_1 and A_2. Conversely we see that this condition is not only sufficient, but also necessary for the point X. We have to find therefore the locus of the vertex of a cone passing through 6 given points. That this is a surface of the fourth order* can be shown as follows.

A quadric surface is fixed by 9 points. Hence through the 6 given points there pass all the quadrics of a linear system with 3 arbitrary parameters:

$$\phi_0 + \lambda_1 \phi_1 + \lambda_2 \phi_2 + \lambda_3 \phi_3 = 0.$$

If X is to be a conical point upon one of these quadrics, 4 conditions must be satisfied, *viz.* the vanishing of the 4 partial derivatives:†

$$\phi_{01}(x) + \lambda_1 \phi_{11}(x) + \lambda_2 \phi_{21}(x) + \lambda_3 \phi_{31}(x) = 0,$$
$$\begin{array}{ccccccccccccc} . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . \end{array}$$
$$\phi_{04}(x) + \lambda_1 \phi_{14}(x) + \lambda_2 \phi_{24}(x) + \lambda_3 \phi_{34}(x) = 0.$$

---

* The "Surface of Weddle." See full discussion in Cayley's *Collected Works*, vol. 7, p. 160 sqq.; and a paper by Chasles, *Comptes Rendus*, vol. 52 (1861) pp. 1157–1162.

† The quadrics ϕ_0, ϕ_1, etc., are supposed to be homogeneous in the coordinates x_1, x_2, x_3, x_4. Partial derivatives are then denoted by added subscripts. Thus the abbreviation $\phi_{21}(x)$ means $\frac{\partial}{\partial x_1} \phi_2 (x_1 x_2, x_3, x_4)$.

Eliminating from these 4 equations, linear in the (x), the parameters λ_1, λ_2, λ_3, we find the equation of a quartic locus for X. Notice that it is the Jacobian of all quadric surfaces that pass through the 6 fixed points.

The intersection of a quadric and a quartic is in general an octavic curve. But this quartic contains entire lines, among them the 15 which join two and two the 6 fixed points; and 3 of these are the directrices a, b, c of the quadric. Hence the proper curve is a quintic, and cuts every directrix in 4 points, but every generator of the quadric (*i. e.* those of the system opposite to a, b, c) in only 1 point. This quintic belongs obviously to the class a_0' of Noether,* a sub-species not particularly described by Halphen,† but included under his second species of quintics. All directrices of the quadric surface are four-fold secants of the curve, so that its projection from any point of the curve upon a plane is a plane quartic with a triple point, proving that the quintic is of deficiency zero, *i. e.* it is rational. Generators of the second system meet it in only one point each, hence it can have no actual double point. Finally, a counting of constants makes it probable that every quintic of Noether's class a_0' can be generated by the above method; if this is true, a geometric proof would disclose an interesting property of the curve.

3. Second Example: Involution of Harmonic Conjugates on Rays Cutting a Conic. Five pairs of conjugate points with respect to a conic completely determine that conic. Hence 5 pairs of points suffice to determine an involution of the second type. But these 5 pairs are not perfectly arbitrary; they are subject to the restriction that the 5 lines joining the respective pairs must have a common point, which we may call a *radiant point*. Any sixth pair of points, in order to belong in the same involution as the first five, must satisfy 2 conditions: first, their joining line must pass through the radiant point; and second, the two points must be conjugate with respect to the same conic which separates harmonically the first 5 pairs. In Fig. 2 O is the radiant point, $A_1 A_2$, $B_1 B_2$, etc., the 5 pairs that determine the involution, o the conic which separates each pair harmonically, and $P_1 P_2$ any sixth pair in the involution.

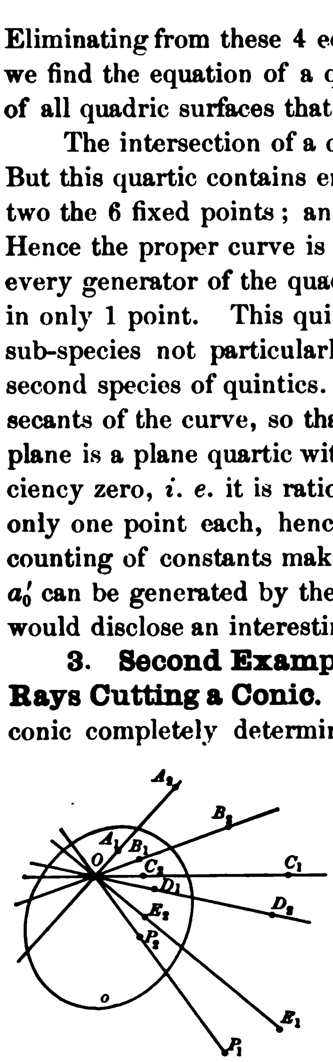

FIG. 2.

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[*] Zur Grundlegung der Theorie d. algebraischen Raumcurven. *Abhndlgn. der kgl. Akad. d. Wissenschaften zu Berlin*, 1882, Anhang, p. 83.

[†] Sur la classification des courbes gauches algébriques. *Jour. de l'Ecole polytechnique*, vol. 52 (1882), p. 162.

To secure a variable center of projection we have therefore to assume in space, not 6 pairs of points taken at random, but one pair on each of 6 directrices of a ruled quadric surface. Then a center of projection X anywhere upon that quadric will give a set of projections satisfying the first condition, for the 6 directrices will project into lines meeting upon the generator that passes through X. As to the second condition, some quadric surface separating harmonically each of the 6 given pairs of points must be a cone having X for its vertex.* Each pair of points, as being conjugates, supplies one linear condition for the coefficients of the equation of that quadric, leaving 3 free parameters $(10 - 6 - 1 = 3)$. Here therefore the reasoning of §2 applies again, and the locus of the vertex X is a quartic surface, the Jacobian of 4 quadrics that have generally no common point. This Jacobian must evidently contain each one of the 12 given points, but need not contain any of their join lines. The curve of intersection is accordingly of order 8 and is of the species a_9 of Noether (*l. c.*, p. 96), or Halphen's 1st species (*l. c.*, p. 165).

By counting constants we may judge that every curve of this species lies on at least one Jacobian surface, and contains 24 pairs of points separated harmonically by all cones of the system. But that these pairs, or any six of them, will lie on as many of the quarti-secants of the curve is apparently improbable. Hence the curve found from six arbitrary pairs of points on six generators of a quadric by the aid of the concept, involution of the second type, must be of a very special kind under its species. To confirm this, it would be needful to study minutely the system of lines joining corresponding points on the general Jacobian surface.

NORTHWESTERN UNIVERSITY,
EVANSTON, ILLINOIS, FEBRUARY, 1902.

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* See Cayley and Chasles, as cited above.

ON SOME CURVES CONNECTED WITH A SYSTEM OF SIMILAR CONICS.

BY R. E. ALLARDICE.

1. Introduction. The problem has been proposed by Steiner[*] of finding the envelope of a system of similar conics circumscribed about a given triangle, and of finding the loci of the centres and foci of the conics of the system. He states that the envelope is a curve of the fourth order having three double points, and gives some of its properties. The problem has been treated by P. H. Schoute in a paper entitled *Application de la transformation par droites symétriques à un problème de Steiner.*[†] In this paper the author discusses the problem of the envelope in detail by a geometrical method, to which reference will be made hereafter, and gives the order of the locus of centres, of the locus of foci, and of the locus of vertices, and the class of the envelope of asymptotes, of the envelope of axes, and of the envelope of directrices. He states, however, that the envelope of the asymptotes is a curve of the sixth class, but does not point out that it breaks up into two curves, each of the third class. I propose to determine analytically the equations of certain of these curves.

2. Definition of Similar Conics by the Use of Asymptotes. It may be taken as a definition or proved as a theorem, according to the point of view, that two conics are similar when the angles between their asymptotes are equal. When rectangular coordinates are used, two conics with their centres at the origin are similar and similarly situated if the terms of the second degree be the same, or proportional, in the two equations. Now these terms of the second degree represent the (real or imaginary) asymptotes, and the angle between these asymptotes is not altered if the conic be displaced in any way. It may easily be shown that the tangent of the angle between the asymptotes of an ellipse is equal to $2abi/(a^2 + b^2)$, where a and b are the semi-axes, and for a real ellipse this quantity is a pure imaginary whose modulus is less than unity. It is obvious that two conics for which this quantity is the same are similar, as the quantity may be expressed in terms of b/a, which in turn can be expressed in terms of the eccentricity.

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[*] Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander (problem 39 of the supplement), *Gesammelte Werke*, vol. 1, p. 446; Vermischte Sätze und Aufgaben, *ibid.*, vol. 2, p. 675.

[†] *Bulletin des sciences mathématiques et astronomiques*, ser. 2, vol. 7 (1883), pp. 314–324.

3. Laguerre's Projective Definition of Angle applied to the Asymptotes. It has been shown by Laguerre and Cayley[*] that the angle between two straight lines can be expressed in terms of the cross-ratio of the range of four points consisting of the points where the straight lines meet the straight line at infinity and the circular points at infinity. More definitely,[†] if α be this cross-ratio and θ the angle between the lines, then $\theta = \frac{1}{2} i \log \alpha$; and from this it may easily be shown that $\tan^2 \theta = -(1 - \alpha)^2 / (1 + \alpha)^2$. Now the infinitely distant points of the two asymptotes are the points where the conic itself meets the line at infinity. We thus have to consider a variable conic that circumscribes a given triangle and meets the straight line at infinity in two points that form with the circular points at infinity a range of constant cross-ratio. But instead of the line at infinity and the two circular points upon it we may take any straight line with two fixed points on it; and these fixed points will be most conveniently assigned as the points of intersection with the given straight line of a fixed conic circumscribing the given triangle. The special case of the system of similar conics will then be obtained by taking the straight line at infinity and the circumscribed circle for the fixed straight line and fixed conic.

4. The Quadric System of Similar Circumscribed Conics. By means of the theory of invariants we may write down the condition that a straight line meet two conics in pairs of points having a given cross-ratio α.[‡]

Let

$$u_x \equiv u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$$

be the given straight line;

$$p_x^2 \equiv p_1 x_2 x_3 + p_2 x_3 x_1 + p_3 x_1 x_2 = 0$$

be the fixed circumscribed conic; and

$$\lambda_x^2 \equiv \lambda_1 x_2 x_3 + \lambda_2 x_3 x_1 + \lambda_3 x_1 x_2 = 0$$

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[*] Laguerre, Note sur la théorie des foyers, *Nouvelles annales de mathématiques*, vol. 12 (1853), p. 64. Cayley, A sixth memoir upon quantics, *Philosophical Transactions*, vol. 149 (1859), pp. 61–90, and *Collected Mathematical Papers*, vol. 2, pp. 561–592. See also Loria, *Teorie geometriche*, where, however, the reference is given to Cayley's fifth, instead of to his sixth memoir.

[†] Compare Klein, *Nicht-Euklidische Geometrie*, vol. 1, pp. 47–60.

[‡] See Clebsch-Lindemann, *Vorlesungen über Geometrie*, vol. 1, p. 281.

be the variable circumscribed conic. The required condition is

$$D'^2 (a-1)^2 - DD'' (a+1)^2 = 0, \qquad (1)$$

where

$$-\tfrac{1}{2} D = \lambda_1^2 u_1^2 + \lambda_2^2 u_2^2 + \lambda_3^2 u_3^2 - 2\lambda_1 \lambda_2 u_1 u_2 - 2\lambda_2 \lambda_3 u_2 u_3 - 2\lambda_3 \lambda_1 u_3 u_1 ;$$

$$-\tfrac{1}{2} D'' = p_1^2 u_1^2 + p_2^2 u_2^2 + p_3^2 u_3^2 - 2p_1 p_2 u_1 u_2 - 2p_2 p_3 u_2 u_3 - 2p_3 p_1 u_3 u_1 ;$$

$$-\tfrac{1}{2} D' = \lambda_1 p_1 u_1^2 + \lambda_2 p_2 u_2^2 + \lambda_3 p_3 u_3^2 - (\lambda_3 p_2 + \lambda_2 p_3) u_2 u_3 - \cdots\cdots .$$

Since according to (1) the parameters λ_1, λ_2, λ_3, which occur linearly in the equation of the conic, are connected by a quadric relation, the conics are said to form a *quadric system.*

5. Locus of Centres of Conics in the Quadric System. Corresponding to the problem of finding the locus of the centres of a system of similar conics we have to find the locus of the pole of u_x with respect to the conic λ_x^2.

If (y_1, y_2, y_3) be the required pole, we have

$$u_1 = \lambda_2 y_3 + \lambda_3 y_2, \text{ etc.},$$

whence, omitting again a factor of proportionality, we have:

$$\lambda_1 = y_1 (u_2 y_2 + u_3 y_3 - u_1 y_1) ; \quad \text{etc.} \qquad (2)$$

Substituting these values of λ_1, λ_2, λ_3 in (1) we get a curve of the fourth order as the locus of the pole of u_x.

If $(\lambda_1, \lambda_2, \lambda_3)$ be regarded as a parametric point, the locus of this point is a system of conics (for varying values of a) of which the equation is given by (1). This system of conics has double contact with the conic $D = 0$ (an inscribed conic), the chord of contact being $D' = 0$. The relations (2) show that this system of conics is transformed into the system of quartic curves by means of a Cremona quadratic transformation. As such a transformation turns a non-specialized conic into a trinodal quartic, we shall expect to find a system of trinodal quartics, all having double contact with the quartic which replaces the conic $D = 0$.

If we substitute the values of λ_1, λ_2, λ_3 in D, D', D'', we find, putting $P_1 = u_1 (p_2 u_2 + p_3 u_3 - p_1 u_1)$, etc.,

$$D = -(u_1 y_1 + u_2 y_2 + u_3 y_3)(u_2 y_2 + u_3 y_3 - u_1 y_1)(u_3 y_3 + u_1 y_1 - u_2 y_2)$$
$$(u_1 y_1 + u_2 y_2 - u_3 y_3),$$

$$D' = P_1 u_1 y_1^2 + P_2 u_2 y_2^2 + P_3 u_3 y_3^2 - (P_1 u_2 + P_2 u_1) y_1 y_2 - \text{etc.},$$

$$D'' = -(P_1 p_1 + P_2 p_2 + P_3 p_3).$$

line u_x in pairs of points in involution; and hence pass through a fourth fixed point.)

For $\theta = 0$, the quartic degenerates into the straight line at infinity and the sides of the triangle of reference. The straight line at infinity corresponds to proper parabolas and the sides of the triangle of reference to degenerate parabolas consisting of pairs of parallel straight lines.

For $\cot^2\theta = -1$, the case of circumscribed conics through either one of the circular points at infinity, the quartic degenerates into two imaginary conics intersecting in the vertices of the triangle and in the circum-centre.

It may easily be shown that the tangents at the vertex A are real when $\cot^2\theta$ is positive, and when $\cot^2\theta$ is negative and $|\cot^2\theta \, \sin^2 A| > 4$. When $4 + \cot^2\theta \sin^2 A = 0$, the tangents coincide and A is a cusp; this value of θ gives the three-cusped hypocycloid when the triangle of reference is equilateral.

7. Similar Conics under the Isogonal Transformation. The transformation by *droites symétriques* employed by P. H. Schoute in the paper referred to above is sometimes called the "isogonal transformation" and consists in substituting $(1/x_1, 1/x_2, 1/x_3)$ for (x_1, x_2, x_3). Schoute proves that to a system of similar circumscribed conics corresponds, in this transformation, the system of tangents to a circle concentric with the circumscribed circle. This theorem is very easily proved by means of the projective definition of an angle. To the straight line at infinity corresponds the circumscribed circle, and *vice versa*; hence the circular points at infinity are transformed into each other by the isogonal transformation. It is easily seen that if the points P, Q, R, S be transformed into the points P', Q', R', S', and if A be any vertex of the triangle of reference, then the cross-ratio of the pencil $A.PQRS$ is equal to the cross-ratio of the pencil $A.P'Q'R'S'$. Now let P_∞ and Q_∞ be the points in which one of a system of similar circumscribed conics meets the straight line at infinity, let P and Q be the corresponding points, namely, the points in which the straight line into which the conic is transformed meets the circumcircle; and let I and J be the circular points at infinity. Then the cross-ratio $(A.IJP_\infty Q_\infty)$ is equal to the cross-ratio $(A.JI PQ)$; thus the latter cross-ratio is constant, and thus the angle PAQ is constant. In other words the chord PQ subtends a constant angle at A and therefore touches a fixed circle concentric with the circumscribed circle.

8. Envelope of Asymptotes. The asymptotes of a conic are the tangents from the pole of the straight line at infinity. Instead of this special line we may in the first instance take any line whatever. It will be convenient to use tangential coordinates; and the equation of the variable circumscribed conic becomes:

$$\lambda_1^2 v_1^2 + \lambda_2^2 v_2^2 + \lambda_3^2 v_3^2 - 2\lambda_1 \lambda_2 v_1 v_2 - 2\lambda_2 \lambda_3 v_2 v_3 - 2\lambda_3 \lambda_1 v_3 v_1 = 0,$$

or

$$\lambda_1 v_1 Q_1 + \lambda_2 v_2 Q_2 + \lambda_3 v_3 Q_3 = 0, \tag{1}$$

where

$$Q_1 = \lambda_2 v_2 + \lambda_3 v_3 - \lambda_1 v_1, \text{ etc.}$$

The equation of the pole of the straight line (u_1, u_2, u_3) is

$$\lambda_1 u_1 Q_1 + \lambda_2 u_2 Q_2 + \lambda_3 u_3 Q_3 = 0. \tag{2}$$

Hence we have to find the envelope of the lines common to (1) and (2), subject to the condition (see (1), §4):

$$D'^2(a - 1)^2 - DD''(a + 1)^2 = 0. \tag{3}$$

Equations (1) and (2) are identically satisfied if we set $\lambda_1, \lambda_2, \lambda_3$ proportional to the cubic functions of the coordinates v:

$$v_1(u_2 v_3 - u_3 v_2)^2, \quad v_2(u_3 v_1 - u_1 v_3)^2, \quad v_3(u_1 v_2 - u_2 v_1)^2.$$

The equation of the envelope, (3), *is then separable into two cubic factors.* These are the following, if we denote as usual $(u_1 v_2 - u_2 v_1)$ by $(uv)_3$, etc.:

$$\begin{aligned} p_1 u_1 (uv)_2 (uv)_3 (u_2 v_3 + u_3 v_2) + p_2 u_2 (uv)_3 (uv)_1 (u_3 v_1 + u_1 v_3) \\ + p_3 u_3 (uv)_1 (uv)_2 (u_1 v_2 + u_2 v_1) \pm h \sqrt{D''}(uv)_1 (uv)_2 (uv_3) = 0, \end{aligned} \tag{4}$$

where h denotes $\dfrac{a+1}{a-1}\cdot$ Since interchange of the two asymptotes to any one conic changes the cross-ratio a into $\dfrac{1}{a}$, or $+h$ into $-h$, it is evident that only one of the two asymptotes touches one of these cubic envelopes, the other asymptote touching the other.

The fact that the envelope of asymptotes resolves into two separate curves might have been anticipated from inspection of equation (3); when $a = 1$ the envelope is given by $D = 0$, but for $a = 1$ this is evidently the product of the equations, each doubly counted, of the infinite points on the sides of the original triangle (or on the sides of the second triangle of reference, since pairs of sides meet at infinity). Hence *D must be itself a square of a rational cubic function of v_1, v_2, v_3,* as seen in (4).

The envelope just now considered,

$$\sqrt{D} = 0,$$

has the line at infinity for a triple tangent. For $a = -1$ the line at infinity is a double tangent, with contacts in I and J.* Accordingly for every value of a or h, each of the envelopes of asymptotes must have the line at infinity for a double tangent and the contacts, if determinate, must be in I and J, two distinct points. Since each envelope is of class 3 and has a double tangent with distinct points of contact, it is of order 4 (like the locus of centres) and has three cusps. And since each side of the original triangle is an asymptote for each value of a, each envelope of asymptotes touches all three sides of the triangle. This fact is obvious also from the equation. From counting of constants we may conclude, what can be proven rigorously by geometric considerations:

Every tricuspidal quartic, touching the three sides of a triangle ABC and touching the line at infinity in the two circular points, is an envelope of asymptotes of similar conics circumscribed to the triangle ABC. All such quartics are three-cusped hypocycloids,† and this fact invites further investigation as to their arrangement in pairs when the triangle ABC is given.

9. The Degeneration of the Sextic Envelope of Asymptotes.

The reason why each asymptote gives a separate envelope, in other words, why the envelope consists of two curves of the third class, and not of a single curve of the sixth class, is that the coordinates of either of the asymptotes may be expressed rationally in terms of those of the other; for, as I have shown in a recent number of the ANNALS,‡ the asymptotes of a circumscribed conic are isotomically conjugate with respect to the triangle of reference. Thus the tangential equation of the envelope of the asymptote $v_1 x_1 + v_2 x_2 + v_3 x_3 = 0$ may be obtained by writing down the condition that this line and the line $a^2 x_1/v_1 + b^2 x_2/v_2 + c^2 x_3/v_3 = 0$, which is the second asymptote of a circumscribed conic having the former line as one asymptote, should intersect at a given angle; and this condition is of the third degree in v_1, v_2, v_3. The envelope of the second asymptote is obtained from that of the first by changing v_1, v_2, v_3 to $a^2/v_1, b^2/v_2, c^2/v_3$.

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* Compare Huntington and Whittemore: Some curious properties of conics touching the line infinity at one of the circular points. *Bulletin Am. Math. Soc.*, vol. 8 (1901), p. 122.

† *Cf.* E. Duporcq, Sur l'hypocycloïde à trois rebroussements, *Nouvelles annales de mathématiques*, ser. 4, vol. 1 (1901), p. 168.

‡ Ser. 2, vol. 2 (1901), p. 148.

NOTE ON MULTIPLY PERFECT NUMBERS.*

BY JACOB WESTLUND.

IN the ANNALS OF MATHEMATICS, ser. 2, vol. 2 (1900/01), p. 103, Dr. D. N. Lehmer proves that no multiply perfect numbers of multiplicity 3, containing less than three distinct primes, exist. In an earlier note on Multiply Perfect Numbers† all numbers of multiplicity 3 of the form $p_1^{\alpha_1}p_2^{\alpha_2}p_3$ were determined. The object of the present note is to determine all numbers of multiplicity 3 of the form $m = p_1^{\alpha_1}p_2p_3p_4$ where p_1, p_2, p_3, p_4 are distinct primes and $p_1 < p_2 < p_3 < p_4$.

Defining a multiply perfect number as one which is an exact divisor of the sum of all its divisors, the quotient being the multiplicity, we have‡ in the present case

$$3 = \frac{p_1 - 1/p_1^{\alpha_1}}{p_1 - 1} \prod_{i=2}^{4} \frac{p_i - 1/p_i}{p_i - 1} = \frac{p_1^{\alpha_1+1} - 1}{p_1^{\alpha_1}(p_1 - 1)} \prod_{i=2}^{4} \frac{p_i + 1}{p_i} \qquad (1)$$

and

$$3 < \prod_{i=1}^{4} \frac{p_i}{p_i - 1}. \qquad (2)$$

From the inequality (2) we infer that $p_1 = 2$ is the only possible value of p_1, since the maximum value of $\prod_{i=1}^{4} \frac{p_i}{p_i - 1}$ will exceed 3 only for $p_1 = 2$. Hence

$$\frac{3}{2} < \prod_{i=2}^{4} \frac{p_i}{p_i - 1},$$

which gives $p_2 < 7$, i. e. the only possible values of p_2 are 3 and 5.

I. Suppose $p_2 = 3$. Then we should have from (1)

$$3 = \frac{2^{\alpha_1+1} - 1}{2^{\alpha_1}} \cdot \frac{4}{3} \cdot \frac{p_3 + 1}{p_3} \cdot \frac{p_4 + 1}{p_4}$$

or

$$9 = \frac{2^{\alpha_1+1} - 1}{2^{\alpha_1-2}} \cdot \frac{p_3 + 1}{p_3} \cdot \frac{p_4 + 1}{p_4}. \qquad (3)$$

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* Read before the Chicago Section of the American Mathematical Society, January 3, 1902.

† Westlund, ANNALS OF MATHEMATICS, ser. 2, vol. 2 (1900/01), p. 172.

‡ *Cf.* Lehmer, *l. c.*

From this we infer that

$$\left(\frac{p_3+1}{p_3}\right)^2 > \frac{9\cdot 2^{a_1-2}}{2^{a_1+1}-1} > \frac{9}{8}$$

or

$$\frac{p_3+1}{p_3} > \frac{3}{4}\sqrt{2}\,.$$

Hence $p_3 < 17$, *i. e.* the only possible values of p_3 are 5, 7, 11, 13.

From (3) we see that $2^{a_1+1}-1$ must be divisible by p_4. Setting $2^{a_1+1}-1 = kp_4$ we have

$$\begin{aligned} 9\cdot 2^{a_1-2}p_3 &= k(p_3+1)(p_4+1) \\ &= (p_3+1)(2^{a_1+1}-1+k) \end{aligned} \tag{4}$$

which may be written in the following form

$$2^{a_1-2}(p_3-8) = (k-1)(p_3+1). \tag{5}$$

Hence we must have $k > 1$ and $p_3 > 8$, and the only possible values of p_3 are 11 and 13. It is easily seen that $p_3 = 13$ does not satisfy (5). For $p_3 = 11$ we get from (5)

$$3\cdot 2^{a_1-2} = (k-1)12$$

or

$$2^{a_1-4} = k-1,$$

and since $2^5\cdot 2^{a_1-4} = kp_4 + 1$, $k = \dfrac{33}{32-p_4}$. Hence $p_4 = 31$, $k = 33$, $a_1 = 9$. The corresponding number is $m = 2^9\cdot 3\cdot 11\cdot 31$ which by trial is found to be multiply perfect.

II. Suppose $p_3 = 5$. In this case we should have

$$3 = \frac{2^{a_1+1}-1}{2^{a_1}}\cdot\frac{6}{5}\cdot\frac{p_3+1}{p_3}\cdot\frac{p_4+1}{p_4} \tag{6}$$

which gives

$$\left(\frac{p_3+1}{p_3}\right)^2 > \frac{5\cdot 2^{a_1-1}}{2^{a_1+1}-1} > \frac{5}{4}$$

or

$$\frac{p_3+1}{p_3} > \frac{1}{2}\sqrt{5}.$$

Hence the only possible value of p_3 is 7. Setting, as in the first case, $2^{a_1+1} - 1 = kp_4$ we get from (6)

$$35 \cdot 2^{a_1-4} = k(p_4 + 1)$$

or

$$3 \cdot 2^{a_1-4} = k - 1,$$

and $k = \dfrac{35}{32 - 3p_4}$. Hence $p_4 < 11$ and since $p_4 > 7$ this is absurd. Therefore p_2 cannot be equal to 5, and the only multiply perfect number of multiplicity 3 of the form $m = p_1^{a_1} p_2 p_3 p_4$ is the number $2^9 \cdot 3 \cdot 11 \cdot 31$.

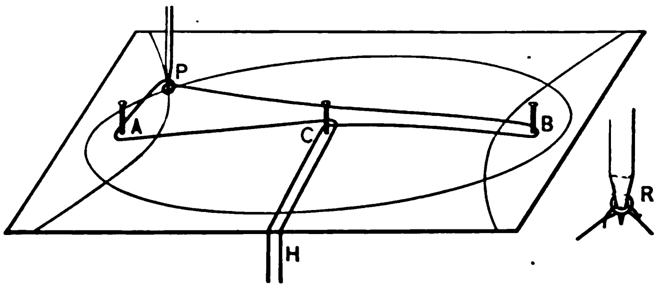
PURDUE UNIVERSITY, DECEMBER 1901.

A MECHANICAL CONSTRUCTION OF CONFOCAL CONICS.

By William R. Ransom.

By the following plan ellipses and hyperbolas whose foci are given may be drawn with very fair accuracy and very great readiness. The method was devised to facilitate plotting in elliptic coordinates, and it enables one to locate points in this system as freely as in the polar and rectangular systems.

Pins are driven at the foci, A, B, and a third at C, which is most conveniently taken midway between A and B. A string is passed loosely around the pencil point and the three pins in the way represented in the drawing. The pencil is then placed at the arbitrarily chosen point P, and the left hand, at H, draws the strings through the fingers until all slack is taken up. All is then ready for drawing either the ellipse or the hyperbola through P. Holding the strings tightly at H, draw them down across the edge of the board, and the pencil traces a branch of the hyperbola down to the axis. Or hold the string tightly against the board between H and C, while the pencil slips laterally in its loop, and we get the ellipse.



The loop about the pencil may be dispensed with in drawing the ellipse, or, in drawing the hyperbola, may be replaced by a small circular link just large enough to slip the lead into, tied loosely, as shown at R, to the two ends of the string severed at P. With a little care, however, the tendency of the pencil to slide in the loop is not great enough to make it necessary to prevent this slipping by inserting such a link.

Tufts College, Massachusetts,
April, 1902.

ON SOPHUS LIE'S REPRESENTATION OF IMAGINARIES IN PLANE GEOMETRY.

BY PERCEY F. SMITH.

THE first published paper of Sophus Lie appeared in the Transactions of the Academy of Christiania in February, 1869, and was concerned with the subject of this article.[*] So great a rôle does this memoir play in the scientific career of the author that a reference to it in the *Leipziger Berichte*, 1897, p. 726, elicits from him the observation that it formed the starting point of all his mathematical investigations.[†] In fact, the point of view there adopted leads in the most natural way to a point-line transformation of space which specialized gives the celebrated line-sphere transformation,[‡] and generalized leads to a new duality defined by two *aequationes directrices* in two sets of point coordinates, upon which is based the monumental memoir in the fifth volume of the *Mathematische Annalen* (1872): Ueber Complexe, insbesondere Linien- und Kugel-Complexe, etc.

I propose to give a presentation of some of the results in the paper referred to above, and I am led to do this not only for the reason that the original lacks clearness and continuity, but also from the fact that Lie's work has been apparently overlooked by writers on this subject.[§] Moreover, I think the method to be expounded possesses merits not shared by such other representations as I have seen. The essential difference consists in this, that Lie assumes the straight line as geometric element in the plane, while other representations are based upon point geometry. Thus the latter give a real line in space as image of a real or imaginary point in the plane,[‖] while Lie's method also leads to a real line in space as representative of a real or imaginary *line* in the plane, each real element moreover being its *own* image. In Lie's representation there is then no change in *kind*.

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[*] The full title is: Repräsentation der Imaginären der Plangeometrie. (Jeder plangeometrischer Satz ist ein besonderer Fall eines stereometrischen Doppel-Satzes in der Geometrie der Linien-Congruenzen). The paper appeared in two parts, pages 16–38, 107–146, the first part, with which we are especially concerned, having previously appeared in *Crelle*, vol. 70 (1869), p. 346. My references are to the Transactions.

[†] *Cf.* also Lie-Scheffers, *Geometrie der Berührungstransformationen*, vol. 1, p. 326, 448.

[‡] First published in the note, Sur une transformation géométrique, *Comptes Rendus*, vol. 71 (1870), p. 579.

[§] *Cf.* Coolidge, A purely geometric representation of all points in the projective plane, *Transactions Amer. Math. Soc.*, vol. 1 (1900), p. 182.

[‖] For a very simple representation *cf.* the memoir by Duport, Sur un mode particulier de représentation des imaginaires, *Annales de l'école normale*, 1880, p. 301.

To all such representations of the imaginaries in Plane Geometry there are of course two objections: *first*, that generalization to three dimensions is futile, and *secondly*, that the content of propositions in the plane is so greatly changed as to be of questionable value. The results of Lie's paper successfully meet the latter objection, I think; while perhaps all that can be said to justify a system such as von Staudt's, which substitutes a real substratum for the imaginaries in three dimensions, is summed up in Lindemann's opinion* that the *recognition of the possibility* of such a representation is a great advantage.† It is, however, right to add that von Staudt's work has led to a well developed theory of involutions.

As to the present presentation as compared with the original, I would say that the definition which I have adopted in §1 is purely projective, and the discussion proceeds in the simplest possible manner therefrom. From this the representation of Lie is deduced in §2 as a metrical special case. The latter, however, in his paper, makes use of an analytic definition only. He there gives, in fact, three different methods of representation, all defined analytically, and it is with the second of these (p. 32 *l. c.*) that we have to do. This is in substance identical with the first, as Lie points out, while the third method is merely defined and not developed. In the second part of the paper referred to, Lie develops synthetically the natural consequences of the representation.

We shall have to deal in our discussion with the four species of imaginary elements in space which present themselves in von Staudt's system, or also by allowing the equations of Analytic Geometry to contain complex coefficients. These are (1) *imaginary points*, each lying on one real line, (2) *imaginary planes*, each containing one real line, (3) *imaginary lines of the first kind*, each lying in one real plane and containing one real point, (4) *imaginary lines of the second kind*, possessing neither a real plane nor a real point.

Since the representation introduces the right line as element of space, we shall have to do with some of the notions of line geometry.‡ Thus in §1 we are led to the configuration of all lines of space intersecting two fixed lines, or, in the language of line geometry, to a linear line congruence of which the fixed lines are the directrices. The middle point of the common perpendicular of the directrices is the *centre*, and one-half of this shortest distance the *constant* of the congruence. I may refer to the text-books quoted for further details.

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* Clebsch-Lindemann: *Vorlesungen über Geometrie*, vol. 2, p. 130. An excellent presentation of von Staudt's theory is given in this volume, page 104.

† Compare also Professor Scott's critique: The status of imaginaries in pure geometry, *Bull. Amer. Math. Soc.*, ser. 2, vol. 7 (1900), p. 163.

‡ Koenigs, *La géométrie reglée.* Plücker, *Neue Geometrie des Raumes, etc.* Clebsch-Lindemann, *l. c.* p. 41.

In von Staudt's system a linear line congruence is the real representative of an imaginary line of the second kind, the congruence consisting of all lines intersecting this line and the conjugate imaginary line.

1. Geometrical Definition of the Representation in Projective Form. Let σ be a real plane in space, the *fundamental plane*, and I an imaginary point, the *fundamental point*, not in σ. Also let L_0, the *fundamental line*, be the real line of the point I, and let S_0, the *fundamental pencil*, be the pencil of real lines in σ whose centre p_0 is the *foot* of the line L_0, that is the point in which L_0 meets σ.

Then any line l in σ may be represented by a real line L in space, viz., the real line of the plane containing I and l.

The line L is uniquely determined save when l belongs to the fundamental pencil, for in that case and in that case only is the plane of I and l real, containing as it does two real lines L_0 and l, and accordingly L *is now any line in a real plane containing the fundamental line.* If l is any other real line in σ, then L will coincide with l.

Conversely, every real line L in space is represented by a line l of σ, viz. the intersection of σ with the plane containing I and L.

The line l is uniquely determined save when L coincides with the fundamental line L_0, for then the plane of I and L becomes indeterminate and l is any line in σ through p_0.

We have therefore established in this way a (1, 1) correspondence between all the lines of σ and the real lines of space, the uniqueness failing only for the lines of the fundamental pencil in σ, and for the fundamental line in space.

It should be noted that l and L intersect in the real point on l.

Consider now in σ the locus of the first class, a point p. The ∞^2 lines of σ through p are represented by the real lines of ∞^2 planes passing through the line Ip. Each of these ∞^2 real lines, however, intersects not only Ip but also its conjugate line $J\bar{p}$, where J and $\bar{p}$ are the conjugate points of I and p respectively. Hence:

Any point p in σ is represented by a real linear line congruence whose directrices are the conjugate lines Ip and $J\bar{p}$.

The line of the congruence lying in σ is the real line through p. It should be remarked also that in general Ip and $J\bar{p}$ are imaginary lines of the second kind.

The congruence becomes special, *i. e.* the directrices intersect, when and only when the real line through p belongs to the fundamental pencil. For, since $\bar{p}$ also lies on this line, Ip and $J\bar{p}$ are now coplanar and accordingly intersect in a real point P. The congruence then consists of all lines through P and all lines in the plane of Ip and $J\bar{p}$, *i. e.* the plane containing P and the fundamental line L_0. Therefore:

A point in σ whose real line belongs to the fundamental pencil is represented by a special congruence consisting of all lines passing through a real point and all lines in the plane determined by that point and the fundamental line.

If p is real, then P coincides with it.

This association at the ∞^3 points of σ, whose real lines belong to the fundamental pencil, and the ∞^3 real points of space is very striking and of great importance in the sequel. The point p_0 in σ is represented by all lines intersecting the fundamental line L_0, *i. e.* by a special linear line complex whose axis is the fundamental line. This complex may be called the *fundamental complex*. The fundamental line L_0 belongs to every congruence representing a point p of σ, for it corresponds to the line $p p_0$.

Consider, now, a pencil of ∞^1 lines in σ through a point p, determined, for example, by a real parameter. Then we have the following:

THEOREM I. *The ∞^1 lines of a pencil in σ whose centre is p are represented in space by the generators of one system of a ruled quadric passing through I and J.*

Proof. The real line in space which represents a line of the given pencil in σ is the intersection of conjugate planes passing respectively through Ip and the conjugate line $J\bar{p}$.

Since the first plane intersects σ in a line of the given pencil, the ∞^1 given lines in σ evidently give rise to two projective pencils of planes through Ip and $J\bar{p}$, respectively, and, by a well known theorem, the line of intersection of corresponding planes of two projective pencils generates a ruled quadric. Finally, Ip and $J\bar{p}$ evidently belong to the generators of the other system, and therefore the quadric passes through I and J.

The quadric degenerates into two flat pencils when the real line through p belongs to the fundamental pencil and to the pencil considered. For, as explained above, in this case the congruence representing p becomes special.

Consider, next, a curve in σ of the second class, a conic C_2. This gives rise in space to a line congruence* of the second order; for through any point P in space will pass two lines of the congruence representing the tangents in σ to C_2 drawn through the foot of IP.

The cone with vertex I passing through C_2 is one of the focal surfaces of this congruence; for if consecutive lines of the latter intersect, then the corresponding consecutive tangents to C_2 do likewise, and this intersection is a point p on C_2; therefore the two lines of the congruence belong to a linear congruence and can intersect only on Ip. The entire focal surface consists of the above cone and the corresponding conjugate cone with vertex at J. The line congruence is then of the second order and fourth class. The real intersection of the focal surfaces, in general a skew curve of order four, corresponds to the points common to C_2 and the fundamental pencil. The tangents of this curve belong to the congruence.

If the conic C_2 passes through the foot p_0 of the fundamental line, then the real intersection of the focal surfaces degenerates into L_0 and a skew curve of the third order passing through I and J. Finally, if the tangents to C_2 through p_0 are real, then it is readily seen that the above cones have double contact on the line which represents the chord of contact of these tangents, and therefore their intersection consists of two conics.

Without prolonging the discussion let it suffice to state that a curve of class n in σ becomes in space a line congruence of order n whose focal surfaces are conjugate cones with vertices I and J. The tangents of the real intersection of these cones belong to the congruence.

2. Lie's Analytical Definition as a Metrical Special Case. Taking XZ as fundamental plane σ, and one of the circular points in the XY plane as I we obtain (using rectangular coordinates) Lie's definition. In fact, if

$$cx = bz + a \qquad (c \text{ real})$$

is any line l in XZ, then the plane through I and l is

$$c(x + iy) = bz + a,$$

and if

$$b = m + in, \qquad a = p + iq,$$

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* In this connection the reader is referred to Picard, *Traité d'analyse*, vol. 1, p. 301.

the real line L of this plane is

$$cx = mz + p, \qquad cy = nz + q.$$

Conversely, given the equations of any real line L of space in this form, we obtain the equation of the corresponding line l in XZ by multiplying the second of these equations by i, adding, and then placing $y = 0$. The plane

$$c(x + iy) = bz + a$$

is real when and only when $c = o$ and b/a is real. Thus we see that the fundamental pencil consists of all real lines in XZ parallel to XX'. The line at infinity in XY is, of course, the fundamental line, as is evidenced by the obvious fact that for $c = o$ the equations of L reduce to those of the line in question, while l becomes z = any constant. J is the other circular point in XY, and p_0 the point at infinity on XX'.

Any line in σ passing through a given point p (α, β) is given by

$$c(x - \alpha) = b(z - \beta),$$

where c and b are arbitrary constants, c being taken real. The plane containing this line and I is then:

$$(1) \qquad c(x + iy - \alpha) = b(z - \beta).$$

This equation defines ∞^2 planes (b complex) through the line Ip

$$(2) \qquad x + iy = \alpha, \quad z = \beta.$$

Let $\alpha = a_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$. Now if $\beta_2 = 0$, *i. e.* if the real line through p belongs to the fundamental pencil, then the line (2) contains the real point $P\,(\alpha_1, \alpha_2, \beta_1)$ and is an imaginary line of the first kind. The real lines of the planes (1), if $c \neq 0$, pass through P; but for $c = 0$ the plane (1) becomes real, and we get all lines of the real horizontal plane through P, $z = \beta_1$.

If however $\beta_2 \neq 0$, (2) is always an imaginary line of the second kind, and the real lines of the ∞^2 planes (1) intersect (2) and also the conjugate line

$$(3) \qquad x - iy = a_1 - i\alpha_2, \qquad z = \beta_1 - i\beta_2.$$

The lines (2) and (3) are accordingly the directrices of the linear congruence formed by all real lines in the planes (1). The shortest distance between them is $2i\beta_2$, and the middle point of this line is (α_1, a_2, β_1), so that we have, adopting the nomenclature of Plücker, THEOREM 2. *In Lie's analytic representation* the imaginary point* $(\alpha_1 + i\alpha_2, \beta_1 + i\beta_2)$ *goes over into the* ∞^2 *lines of a linear line congruence whose centre is* $(\alpha_1, \alpha_2, \beta_1)$ *and constant* $i\beta_2$.

The first method given by Lie in the paper referred to represents the imaginary point $(\alpha_1 + i\alpha_2, \beta_1 + i\beta_2)$ by the real point of space $(\alpha_1, \alpha_2, \beta_1)$, to which is assigned a *weight* β_2. He very soon points out (page 20) that β_2 measures the constant of a line congruence, but adheres by preference to this representation throughout the paper. The ∞^3 points of the fundamental plane for which $\beta_2 = 0$, *i. e.* the points whose real lines constitute the fundamental pencil, he calls *null points*, which name I shall adopt. The important point is this, that each null point is represented by a special congruence whose centre is a real point P and whose plane is the plane PL_0. We may say, indeed, that the image of the null point is P itself, and conversely the special congruence constructed as aforesaid with P leads back to the original null point. Ambiguity enters only for the point p_0 which gives any point in the fundamental line.

THEOREM 3. *Lie's representation establishes a* (1, 1) *involutory correspondence between the lines of the plane and the real lines of space in which the fundamental configurations are, in the plane a pencil of real lines, and in space a line through the vertex of this pencil. By this is also set up a* (1, 1) *involutory correspondence of the* ∞^3 *points on the lines of this pencil (the null points of the plane), and the real points of space, and the uniqueness fails only for the vertex of the pencil.*

In this theorem are set forth the peculiar merits of Lie's method to which I have already referred.

It is to be observed that the combination of null point p and line containing p is represented by a real point P and a real line through P. Also the range of null points on any line l is projective with the real range on the corresponding line in space.

3. Correlation in the Plane and the Corresponding Point-Line Transformation of Space. On page 35 of Lie's paper is introduced the question of what arises in space when the representation is carried out on a correlation in the plane. The answer is immediate, for since the ∞^3 nulpoints of σ go over into ∞^3 lines, then we obviously have a transformation T of the ∞^3 points of space into an assemblage of ∞^3 lines, *i. e.* a line complex

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* *Lie, l. c.* p. 33.

Γ. The general features of T we may deduce by considering a polar reciprocation in σ with respect to a conic C_2 (Lie, §27). Since if p in σ reciprocates into l, then any null point of l corresponds to a line through p, we may state one fundamental property of T thus:

THEOREM 4. *If a point P in space describes a line L of the complex Γ, the corresponding line will turn around the point corresponding to L, i. e. will generate a cone whose vertex is that point.*

We next determine the degree of this cone (which we shall call a *complex cone*), which will give us the *order* of the line complex. If P describes an arbitrary line L the corresponding ∞^1 lines of Γ will represent the ∞^1 lines of a pencil in σ, *i. e.* by Theorem 1, will be generators of one system of a ruled quadric through I and J. Hence

THEOREM 5. *The complex cones of Γ are quadric cones through I and J, and therefore Γ is a line complex of order 2.*

Consideration of the relation of C_2 to the fundamental pencil in σ will lead to the singular properties of Γ. In the general case, p_0 reciprocates into a line l_0 which corresponds to a line L'_0 of Γ. Then, as P describes L'_0, the corresponding complex cone must degenerate into two planes; for the vertex must correspond to the base point p_0, *i. e.*, is any point of IJ. We assume this plane-pair *real*,* say E_1 and E_2, intersecting σ in l_1 and l_2. Now the ∞^3 lines of Γ are grouped in ∞^1 linear congruences representing the range on l_0 into which the lines of the fundamental pencil reciprocate. Two points of this range are null points, viz. the intersections of l_0 with l_1 and l_2; hence two *special* congruences belong to Γ; their centres are the intersections of L'_0 with E_1 and E_2, and their planes are E_1 and E_2. If A_1, A_2 are the points of intersection of L'_0 with E_1, E_2, then A_1 transforms into any line in E_2 while any point in E_2 gives a line through A_1. And since an arbitrary line L in space intersects E_1 and E_2, as P describes L the ruled quadric generated by the corresponding complex line passes through A_1 and A_2. The complex cones then also contain these points. We then easily obtain the following definition of Γ:

Given the tetrahedron A_1A_2IJ and a complex line L, then Γ consists of the generators of the ∞^2 ruled quadrics circumscribed to the tetrahedron and containing L, which belong to the same system as L.

For if p is the pole of l, and l' any line through p, then the ∞^1 null points on l' give the generators of such a quadric; and all lines of Γ are determined by taking for l' any line through p. Since the definition is symmetrical with respect to the vertices of the tetrahedron, we see that every line through each vertex and in each face of the tetrahedron belongs to Γ. The complex Γ is then known to be a *tetrahedral complex.*[*] The complex lines intersect the faces of the tetrahedron $A_1 A_2 IJ$ in a constant cross ratio.

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* This assumption is made in order to use our representation.

Theorem 6. *The general polar reciprocation in the fundamental plane yields, under Lie's representation, a transformation of the* ∞^3 *points of space into the lines of a tetrahedral complex, the points* I *and* J *being vertices of the fundamental tetrahedron.*[†]

The complex Γ will degenerate and the transformation take on special properties if we assume a special relation of C_2 and the fundamental pencil. For example, let the point p_0 be on C_2. Then it is readily seen that the complex has two singular points I, J and two singular planes intersecting in IJ. These planes intersect σ in the tangent to C_2 at p_0 and in the conjugate line. Without discussing the corresponding transformation, consider next what arises if p_0 reciprocates into a line l_0 belonging to the fundamental pencil. Then Γ degenerates into two special linear complexes, whose axes are the line IJ and a line in the plane IJl_0. The corresponding transformation of space becomes very simple. This case, however, as well as others of particular interest, appear if we do not limit ourselves to a reciprocation.[‡]

Consider then a correlation in σ such that p_0 transforms into a line l_0 of the fundamental pencil. Then a second line l'_0 (in general imaginary) through p_0 will transform back into p_0. In the first instance the fundamental pencil correlates into a range on l_0 not including p_0, and the corresponding complex in space consists of ∞^1 special linear congruences whose centres lie upon a non-degenerate conic through I and J in the plane of IJ and l_0. That is, *the complex* Γ *is now a special quadratic complex consisting of all secants of a conic through* I *and* J. On the contrary the range determined on l'_0 dual with the fundamental pencil contains one null point p_0, and accordingly the complex arising in space degenerates into two linear complexes one of which is the special complex whose axis is IJ, *i. e.* the fundamental complex. Ambiguity in space is best avoided by regarding the transformation as a duality between two spaces r and R, such that the points of each correspond to the lines of a line complex in the other, these complexes being in R all secants of a fixed conic, and in r a general linear complex and the fundamental complex. We note here the essential characteristics of the line-sphere transformation; for the range on an arbitrary line in r must give rise to a ruled quadric containing the fixed conic, and if the latter is the imaginary circle at infinity, the quadric becomes a sphere.*

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[*] The reader is referred to Lie-Scheffers, *l. c.* p. 311, for an excellent discussion of this complex.

[†] This transformation was first remarked by Reye in 1868. *Cf. Geometrie der Lage*, zweite Abteilung, p. 125.

[‡] Lie discusses only the case of a reciprocation, and thus the line-sphere transformation did not appear until later, as already remarked.

If we require that l_0' also shall belong to the fundamental pencil, then the range on l_0 does contain p_0, and we readily see that the complex in either R or r degenerates into two special complexes, one of which is the fundamental complex.

The consideration of other special cases would not be without interest. Those enumerated may suffice in this place, however, and are moreover particularly signalled by Lie in all references to this subject.†

4. Duality of Space Defined by Two Bilinear Equations.

Writing the equation of a correlation in XZ in the form

$$(4)\quad (Ax + Bz + C)X + (A'x + B'z + C')Z + A''x + B''z + C'' = 0,$$

or also

$$(5)\quad (AX + A'Z + A'')x + (BX + B'Z + B'')z + CX + C'Z + C'' = 0,$$

and confining ourselves to *null points*, *i. e.* to real values of z and Z, then writing $x + iy$ and $X + iY$ for x and X respectively, and separating (4) into real and imaginary parts, we obtain

$$(6)\qquad (a + ia')\,(X + iY) + (c + ic')\,Z + b + ib' = 0, \text{ or}$$

$$(7)\qquad \begin{cases} aX - a'Y + cZ + b = 0 \\ a'X + aY + c'Z + b' = 0, \end{cases}$$

that is since a, a', b, b', c, c' are *linear* in x, y, z, *two bilinear equations*. Denoting space by r or R according as we represent a point by x, y, z, or X, Y, Z, we may state

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* Any point of R not on the fixed conic corresponds in r to a line of the general linear complex. But a point on that conic gives all lines in a plane through the axis of the fundamental complex. This representation of the general linear complex upon point space was first noticed by Noether, *Göttinger Nachrichten*, 1869, p. 305.

† *Cf. e. g. Mathematische Annalen*, vol. 5 (1872), p. 165; *Leipziger Berichte*, 1897, p. 728.

Theorem 7. *Lie's representation applied to a correlation in the plane leads to a duality in space defined by two bilinear equations.*

Plücker discussed the duality in space defined by one *aequatio directrix*, but it remained for Lie to extend this notion to two equations, and the discovery came about precisely in the manner outlined above.*

The discussion of (7) is very simple, and the properties of the transformation deduced above are very easily established. The line $a = 0$, $a' = 0$ is L_0, and its points of intersection with the ruled quadric

$$\frac{b}{c} = \frac{b'}{c'}$$

are A_1 and A_2, and each of the points gives in R a plane $Z = \text{const.}$ Furthermore, the point I in R corresponds to the *plane* $a - ia' = 0$, *i. e.* the plane $A_1 A_2 J$ in r, etc.

Turning now to special cases, suppose p_0 in r, *i. e.* the point at infinity on XX', gives a line through that point itself. For this it is necessary and sufficient that $A = 0$. The line $a = 0$, $a' = 0$, now becomes the line at infinity in XY, and the singular tetrahedron reduces to IJ and two planes parallel to XY.

Suppose, however, that p_0 in r gives a real line l_0 through it; then from (5), A'/A'' must be *real.* And as we may take, without loss of generality, the line at infinity in XZ for l_0, A' becomes zero and (4), (5), and (7) reduce to

$$(8) \qquad (Bz + C)X + (B'z + C')Z + A''x + B''z + C'' = 0$$

$$(9) \qquad A''x + (BX + B'Z + B'')z + CX + C'Z + C'' = 0,$$

$$(10)\cdot \qquad \begin{cases} (a_0 z + a_1)X - (a'_0 z + a'_1)Y + (c_0 z + c_1)Z + b = 0 \\ (a'_0 z + a'_1)X + (a_0 z + a_1)Y + (c'_0 z + c'_1)Z + b' = 0, \end{cases}$$

where b and b' are still linear in x, y, z.

Taking now any line in r, $x = mz + p$, $y = nz + p$, substituting for x and y in (10), and eliminating z, we obtain a quadric whose trace upon the plane at infinity is readily found to be given by

$$(11) \qquad \begin{aligned} &(a_1 a'_0)(X^2 + Y^2) + (c_1 c'_0)Z^2 + [(a_0 c_1) + (a'_0 c'_1)]\,YZ + \\ &\qquad [(a'_0 c_1) + (a_1 c'_0)]'XZ = 0, \end{aligned}$$

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* *Cf. Mathematische Annalen*, vol. 5 (1872), p. 143-157.

in which the parentheses denote determinants. This conic obviously passes through the circular points I and J.

The constants in (8) may be specialized with no loss of generality so that

$$B = i, \quad C = -1, \quad B' = -i, \quad C' = -1, \quad A'' = -1, \quad B'' = C''' = 0.$$

In fact, this amounts to choosing (0, 0, 0), (0, 0, 1), and (1, 0, 0) in R to represent in r the axis ZZ' and the two minimum lines in xz passing through (– 1, 0, 0). We thus obtain from (10) the equations

$$(12) \qquad \begin{cases} X + zY + Z + x = 0, \\ zX - Y - zZ - y = 0, \end{cases}$$

while the conic (11) becomes

$$(13) \qquad X^2 + Y^2 - Z^2 = 0.$$

To obtain then from (12) the line-sphere transformation it suffices to replace Z by iZ, and these become

$$(14) \qquad \begin{cases} X + iZ + x + zY = 0, \\ z(X + iZ) - y - Y = 0. \end{cases}$$

These equations agree essentially with those adopted by Lie[*] by merely interchanging Z and Y.

From the equations (9) we see that the point at infinity on XX' in R gives in r an imaginary line in XZ when B/C is not real. The discussion of the previous section shows then that the line complex in r degenerates into a general linear complex and the special complex whose axis is IJ. We may readily verify this from the equations (12), for solving these for x and y, we have

$$(15) \qquad \begin{cases} x = -Yz - (X + Z), \\ y = (X - Z)z - Y. \end{cases}$$

Comparing these with the equations $x = rz + \rho$, $y = sz + \sigma$ as written by Plücker, we find

$$r = -Y, \quad \rho = -(X + Z), \quad s = X - Z, \quad \sigma = -Y,$$

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[*] A detailed study of the line-sphere transformation is given in Lie-Scheffers, *Berührungstransformationen*, chap. 10, p. 411.

and for the remaining line-coordinate,*

$$\eta \equiv r\sigma - \rho s = X^2 + Y^2 - Z^2.$$

Then any point (X, Y, Z) not at infinity on the cone $X^2 + Y^2 - Z^2 = 0$ corresponds to a unique line of the general linear complex

$$r - \sigma = 0,$$

but the points excluded lead to the special linear complex

$$\eta = 0.$$

Finally, if the ratio B/C in (8) is real, $a_1 a_0' - a_0 a_1' = 0$, and the conic defined by (11) and the plane at infinity degenerates into two lines one of which lies in the plane XY; *i. e.* the line complex in either R or r consists of two special complexes one of which is the fundamental complex. As before, any point in R not on the degenerate conic at infinity corresponds to a line of the special complex in r whose axis is different from IJ. An example of this case is afforded by the polar reciprocation

$$(16) \qquad zZ + X + x = 0,$$

which corresponds to the transformation of space

$$(17) \qquad \begin{cases} X + x + zZ = 0, \\ Y + y = 0. \end{cases}$$

Any point not at infinity in XY or XZ corresponds to a line parallel to the plane XZ.

Lie remarks that the transformation (17) is identical with that made use of by Euler and Ampère in the theory of partial differential equations of the first order, viz.

$$X = p, \quad Y + y = 0, \quad Z + z + px = 0, \quad p = \frac{\partial z}{\partial x},$$

and in fact (17) come from these by elimination of p.

Summing up, then, we may state

THEOREM 8. *The transformation of space established by two bilinear equations in the variables (x, y, z) and (X, Y, Z) leads to a duality of the spaces r and R in which the points of either correspond to the lines of a line complex in the other. In the general case, these complexes are general tetra-*

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* *Cf.* Clebsch-Lindemann, *l. c.* p. 44.

hedral complexes. The special cases (1) *when the complex in* r *degenerates into a general and a special linear complex and that in* R *into all secants of a conic, and* (2) *when the complexes in both spaces degenerate into linear complexes, present themselves naturally and are among the most important of the special cases.*

Of course, by a projective transformation in either r or R, or in both, the equations (7) assume the general bilinear form. A discussion of all special cases would seem to be not without interest.

5. Point-Line Transformation in General. In 1871 Lie attacked the problem of the determination of all algebraic transformations of space such that all points go over into the lines of a line complex, and the lines of a line complex into the points of space. He was unable, at that time, to solve the problem completely, and contented himself with the remark that all* (1, 1) transformations are given by bilinear equations. The solution was finally given in his paper Liniengeometrie und Berührungstransformationen, *Leipziger Berichte*, 1897, p. 687, and the results are recapitulated on page 740. It turns out that the only cases in addition to those defined as above are:

(1) The complex in r is a general linear, and in R a special quadratic complex consisting of all the tangents of a general quadric.

(2) Both complexes are special and consist of the tangents of developable surfaces, and one of these (or both as above) may become a special *linear* complex.

The first case arises in the simplest possible manner by using a point transformation given by Darboux† by which the secants of a conic become the tangents of a general quadric. This is done by taking the case discussed above of a general linear complex in r‡ and the special complex in R of all secants of a conic, and transforming R by Darboux's point transformation.

Examples of the second type are derived by setting up a (1, 1) correspondence between the generating planes of two developables, and then assuming a duality such that a line in one plane shall correspond to a point in the other. The duality may be established, for example, by taking the conjugate of the given line with respect to a quadric, and the point of intersection of this conjugate with the corresponding plane of the other developable. In this way is obviously established a correspondence of the points of space and the tangents to a developable such that point and tangent through it go over into a like combination.

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* With this exception, that the general case when both complexes are linear and special is not completely represented. *Cf.* Lie, *Mathematische Annalen*, vol. 5 (1872), p. 167, footnote.

† Darboux, *Leçons sur la théorie générale des surfaces*, vol. 3, p. 493.

‡ The fundamental complex may be omitted in the statement for it arises from the ∞^1 points of the fundamental conic in R.

It is to be remarked that *both* complexes are general in one case only, viz. when both are tetrahedral complexes.

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NOTE ON THE GROUP OF ISOMORPHISMS OF A GROUP OF ORDER p^m.

By G. A. Miller.

The first part of the following note is devoted to a study of some of the properties of the holomorphisms of a group of order p^m, p being any prime, which correspond to operators whose order is a power of p in the group of isomorphisms. In the second part an abelian subgroup of the group of isomorphisms of any abelian group of order p^m is determined. It is proved that this abelian subgroup is one of a series of conjugate subgroups which have in common the invariant operators of the group of isomorphisms.

1. Let P represent any group of order p^m, p being any prime, and let $P_1, P_2, \ldots . P_m$ represent any series of subgroups of orders $p, p^2, \ldots . p^m$ respectively such that P_{a-1} is contained in P_a, $a = 2, 3 \ldots . m$. The main object of this note is to consider all the holomorphisms of P which can be obtained on condition that every operator of P_a which is not in P_{a-1} corresponds to itself multiplied on the left by some operator of P_{a-1}.[*] It will first be proved that all such holomorphisms of P correspond to operators of order p^λ in the group of isomorphisms (I) of P; and, conversely, that each of the holomorphisms of P which corresponds to an operator of order p^λ in I is of this form.

If t_1, t_2 are any two operators of I which correspond to two such holomorphisms, then must $t_1 t_2$ have the same property. That is, to the totality of the possible holomorphisms for any series of subgroups such as $P_1, P_2, \ldots . P_m$ there corresponds a subgroup (I_1) of I. Let t_3 be any operator of I_1. In the holomorphism which corresponds to t_3 some operator (s) of P corresponds to $s_1 s$, where s_1 is commutative[†] with t_3. From this it follows that $t_3^{-n} s t_3^n = s_1^n s$, and hence the order of t_3 must be a power of p. Since t_3 is any operator of I_1 it follows that the order of I_1 is a power of p. When P is abelian this result may also be obtained by means of the known formula[‡]

$$t^{-n} s_a t^n = s_{a+n} s_{a+n-1}^n \cdot \cdot \cdot \cdot \cdot s_{a+n-r}^{\frac{n(n-1) \ldots (n-r+1)}{r!}} \cdot \cdot \cdot \cdot s_{a+1}^n s_a$$

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[*] *Cf.* Burnside, *Theory of Groups of Finite Order*, 1897, p. 249.

[†] In the substitution group of degree p^m, determined by the two groups P and I. *Cf.* Burnside, *l. c.*, *p.* 227.

[‡] *Bulletin of the American Mathematical Society*, vol. 7 (1901), p. 351.

whenever

$$t^{-1}s_\beta t = s_{\beta+1}s_\beta, \qquad \beta = a,\ a+1,\ \cdot\cdot\cdot\cdot\cdot\ a+n-1\,;$$

for s_{a+n} is the identity if $n > m - 1$, and n may be so chosen that each of the exponents

$$n,\ \cdot\cdot\cdot\cdot\cdot\ \frac{n(n-1)\ \cdot\cdot\cdot\cdot\ (n-r+1}{r\,!},\ \cdot\cdot\cdot\cdot\cdot\ n$$

is divisible by any power of p.

Frobenius has proved that in a group (P') of order $p^{m'}$ the total number of subgroups P'' of order $p^{m''}$ ($m'' < m'$) is $\equiv 1 \bmod p$. If P' is an invariant subgroup of our main group P of order p^m, a group P'' is invariant either under P or under one of a set of subgroups P'' conjugate under P. As the total number of these conjugates must be a multiple of p, it follows that the number of subgroups of order $p^{m''}$ which are contained in P' and invariant under P is $\equiv 1 \bmod p$.

We consider now any subgroup $\bar{I}$ of I, the order of $\bar{I}$ being $p^{\bar{n}}$, and prove that it is a subgroup I_1 related in the way explained in the first paragraph to at least one series of subgroups $P_1, P_2, \cdot\cdot\cdot\ P_m = P$ of P, in which, furthermore, every subgroup P_a is invariant under P. The group I_1 connected with an arbitrary series of subgroups is such a group $\bar{I}$, and it is connected also with a series of subgroups of the particular character just specified.

The subgroup $\bar{I}$ of I leaves invariant $P = P_m$, and at least one of its invariant subgroups (P_{m-1}) of order p^{m-1}, since its order is a power of p and the number of such subgroups is $\equiv 1 \bmod p$. Similarly it leaves invariant at least one (P_{m-2}) of the subgroups of P_{m-1} which are of order p^{m-2} and invariant under P_m. And so on. Thus the group $\bar{I}$ does leave invariant each one of a series of subgroups $P_1, \ldots P_m$ of the kind specified. But further it leaves it invariant in the way specified in the first paragraph, as one readily proves; for the quotient group P_{a+1}/P_a is of order p and its group of isomorphisms is of order $p - 1$, which is prime to the order of $\bar{I}$. Such subgroups as I_1 depend, in general, upon P_1 and also upon the manner of selecting P_2, $P_3, \ldots, P_{m-1}$ after P_1 has been chosen. In particular, when P is cyclic, these subgroups can be chosen in only one way, while they can be chosen in a number of ways depending upon m when P is abelian and of type $(1, 1, 1, \ldots)$. In the latter case the totality of the holomorphisms for one series of subgroups such as $p_1, p_2, \ldots\ldots p_m$ is evidently the same as that for any other series, so that I_1, which is of order $p^{\frac{m(m-1)}{2}}$, has just as many conjugates under I as there are different ways of selecting such a series; viz. $(p^m - 1)(p^{m-1} - 1) \cdot\cdot\cdot\cdot\cdot (p - 1) \div (p - 1)^m$. Each of these conjugates is therefore invariant in a subgroup (I_2) of I, whose order is $p^{\frac{m(m-1)}{2}}(p - 1)^m$. Moreover, the quotient group I_2/I_1 is the direct product of m cyclic groups.

In the last example it was observed that, if any of the subgroups P_1, P_2, $\ldots\ldots P_{m-1}$ is replaced by a different one, the corresponding subgroups of I will be conjugate, but not identical with I_1. This is clearly always the case when a holomorphism of P may be obtained by multiplying an operator of P_α by an arbitrary operator of $P_{\alpha-1}$, $\alpha = 2, 3, \ldots\ldots m$. When the last condition is satisfied, let I_2 represent the largest subgroup of I in which I_1 is invariant. We proceed to prove that I_2/I_1 is always a subgroup of the direct product of cyclic groups of order $p - 1$.

In all the holomorphisms which correspond to I_2, each of the subgroups $P_1, P_2, \ldots\ldots, P_m$ corresponds to itself, and conversely, if each of these subgroups corresponds to itself in any holomorphism of P, this holomorphism must correspond to some operator of I_2. In all these holomorphisms the operators of $P_\alpha/P_{\alpha-1}$ $(\alpha = 2, 3, \ldots m)$ correspond to some power of themselves. Let t_4, t_5 be any two operators of I_2 and consider the holomorphism which corresponds to the commutator of $t_4^{-1} t_5^{-1} t_4 t_5$. Since all the operators of $P_\alpha/P_{\alpha-1}$ must correspond to themselves in this holomorphism, it follows from the preceding paragraph that the order of $t_4^{-1} t_5^{-1} t_4 t_5$ is a power of p. Hence I_1, which is composed of all the operators of I whose orders are powers of p, must include all the commutators of I. As the quotient group with respect to any invariant subgroup which includes the commutator subgroup is abelian,[*] I_2/I_1 must be abelian.[†] Furthermore, since the groups $P_\alpha/P_{\alpha-1}$ are of order p and correspond to themselves in all these holomorphisms, I_2/I_1, *must be included in the direct product of cyclic groups of order* $p - 1$.

It may be of interest to observe that a change in the series of subgroups $P_1, P_2, \ldots\ldots P_{m-1}$ does not necessarily affect I_1. For instance, when P is the direct product of two cyclic groups (C_1, C_2) of orders p^{m-1}, p respectively $(m > 2)$, its group of isomorphisms (I) is of order $p^m (p - 1)^2$.[‡] In this case, let C_1 equal P_{m-1}. This determines the series $P_1, P_2, \ldots P_{m-1}$ and the corresponding I_1 is clearly of order p^{m-1}. The subgroup I_1 includes all the operators of I which satisfy the following conditions: the orders are powers of p, and they transform each of the cyclic subgroups of order p^{m-1} in P into itself. When P_{m-1} is replaced by any other cyclic subgroup of the same order, the remaining subgroups of the series $P_1, P_2, \ldots . P_{m-1}$ will not be changed, and the corresponding subgroup of I clearly satisfies the same condition as before, and hence it is identical with I_1.

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[*] *Quarterly Journal of Mathematics*, vol 28, 1896, p. 267.

[†] Wendt, *Mathematische Annalen*, vol. 55, 1901, p. 480.

[‡] *Cf. Transactions of the American Mathematical Society*. vol. 2, 1901, p. 260.

2. In what follows it will be assumed that P is abelian. If p^{a_1} is the highest order of an operator in P, then it is possible to obtain $p^{a_1-1}(p-1)$ distinct holomorphisms of P by raising each one of its operators to the same power. It is known that these holomorphisms correspond to the $p^{a_1-1}(p-1)$ invariant operators of I.* We proceed to consider an important abelian subgroup of I which includes the characteristic subgroup composed of these invariant operators.

Let $H_1, H_2, \ldots . . H_n$ be any set of independent generating cyclic subgroups of P whose orders are $p^{h_1}, p^{h_2}, \ldots . p^{h_n}$ respectively; and consider any holomorphism of P in which each of these subgroups corresponds to itself. It is clearly possible to establish an arbitrary holomorphism of one of these subgroups with itself without affecting the holomorphism of any one of the other subgroups. Hence it follows that the totality of the holomorphisms of P in which each of these subgroups corresponds to itself must correspond in I to the direct product (A) of n cyclic groups of orders

$$p^{h_1-1}(p-1), \quad p^{h_2-1}(p-1), \quad \ldots . . \quad p^{h_m-1}(p-1)$$

respectively, whenever $p > 2$. When $p = 2$, the subgroup A is the direct product of a group of order 2^n and of type $(1, 1, 1) \ldots . .$ and n cyclic groups of orders $2^{h_1-2}, 2^{h_2-2}, \ldots . . 2^{h_m-2}$ respectively. The only case in which A reduces to the identity is when P is of type $(1, 1, 1 \ldots . .)$ and $p = 2$.

Let $S_1, S_2, \ldots . . S_n$ represent a set of generators of the cyclic subgroups $H_1, H_2, \ldots . . H_n$ respectively, and let $H'_1, H'_2, \ldots . H'_n$ represent a second set of independent generating cyclic subgroups of P. At least one of the latter subgroups (H'_a) is not generated by a single one of the operators of $S_1, S_2, \ldots . . S_n$. A generator of H'_a is therefore of the form $S_a^{a_1} S_\beta^{\beta_1} \ldots . . . .$, where at least two of the exponents $a_1, \beta_1, \ldots .$ differ from zero. As the subgroup A $(p > 2)$ includes some operators which transforms H'_a into itself, multiplied by some operator which is not found in H'_a, it follows that A *transforms into itself each member of only one of the possible sets of independent generating cyclic subgroups of* P, *whenever* $p > 2$.

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* *Cf.* the last foot-note.

From the preceding paragraph it follows that A has as many conjugates under I as there are different combinations of generating subgroups of P, whenever p is odd. In this case I contains no operators that transform A into itself besides those of A and those which transform the totality of the subgroups $H_1, H_2, \ldots . H_n$ into itself, but permute some of them. The latter operators exist only when at least two of the independent generators of P are of the same order. Moreover, P contains no operator besides the identity which is invariant under A.

When $p = 2$, all the operators of order 2 in P are invariant under A, and hence A reduces to the identity when P is of type $(1, 1, 1, \ldots)$, as was observed above from another point of view. Since all the operators of A do not transform into itself any operator of P whose order exceeds 2, they cannot transform each of the subgroups $H'_1, H'_2, \ldots . H'_n$ into itself unless the order of no more than one factor in the product $S_\alpha^{\alpha_1} S_\beta^{\beta_1} \ldots .$ exceeds 2 for every H'_α. This condition is clearly sufficient as well as necessary.

All the conjugates of A have the $p^{a_1-1}(p-1)$ invariant operators of I in common for all values of p, since each of these operators transforms every subgroup of P into itself.

Moreover, it is easy to prove that every operator (t) which is found in all the conjugates of A is also included among these invariant operators of I. From the fact that the product $S_1 S_2 \ldots S_n$ may be used as an independent generator of P it follows that

$$t^{-1} S_1 S_2 \ldots . S_n t = (S_1 S_2 \ldots S_n)^\beta \quad \text{and} \quad t^{-1} S_i t = S_i^{\beta_i} \quad i = 1, 2, \ldots, n.$$

Hence

$$(S_1 S_2 \ldots S_n)^\beta = S_1^{\beta_1} S_2^{\beta_2} \ldots S_n^{\beta_n}.$$

We may therefore set $\beta_1 = \beta_2 = \cdots = \beta_n \, \beta$. Since t transforms each generator of P into its βth power, it also transforms each operator of P into this power; that is, *the* $p^{a_1-1}(p-1)$ *invariant operators of* I *are the only ones which are common to all the conjugates of* A *under* I.

STANFORD UNIVERSITY,
JANUARY, 1902.

EVALUATION OF SLOWLY CONVERGENT SERIES.*

BY L. D. AMES.

1. General Theory. Any convergent series

$$S = \tfrac{1}{2}u_1 + \tfrac{1}{2}u_2 + u_3 + \cdots \tag{1}$$

can be written in the form:

$$S = \tfrac{1}{2}u_1 + \tfrac{1}{2}[(u_1 + u_2) + (u_2 + u_3) + \cdots]. \tag{2}$$

Repeating the process on the series in brackets we have:

$$S = \tfrac{1}{2}u_1 + \tfrac{1}{4}(u_1 + u_2) + \tfrac{1}{4}[(u_1 + 2u_2 + u_3) + (u_2 + 2u_3 + u_4) + \cdot\cdot], \tag{3}$$

$$\begin{aligned} S = {} & \tfrac{1}{2}u_1 + \tfrac{1}{4}(u_1 + u_2) + \tfrac{1}{8}(u_1 + 2u_2 + u_3) \\ & + \tfrac{1}{8}[(u_1 + 3u_2 + 3u_3 + u_4) + (u_2 + 3u_3 + 3u_4 + u_5) + \cdot\cdot], \end{aligned} \tag{4}$$

and in general:

$$\begin{aligned} S = {} & \frac{1}{2}u_1 + \frac{1}{2^2}(u_1 + u_2) + \cdots\cdot \\ & + \frac{1}{2^k}\left[u_1 + (k-1)u_2 + \frac{(k-1)(k-2)}{1\cdot 2}u_3 + \cdots + u_k\right] + R_k, \end{aligned} \tag{5}$$

where

$$\begin{aligned} R_k = {} & \frac{1}{2^k}\Big[(u_1 + ku_2 + \frac{k(k-1)}{1\cdot 2}u_3 + \cdots\cdot + u_{k+1}) \\ & + (u_2 + ku_3 + \frac{k(k-1)}{1\cdot 2}u_4 + \cdots + u_{k+2}) + \cdots\Big]. \end{aligned} \tag{6}$$

It will be proved in §3 that

$$\lim_{k=\infty} R_k = 0.$$

Assuming this for the present we see that (1) may be written in the form:

$$S = \frac{1}{2}u_1 + \frac{1}{2^2}(u_1 + u_2) + \frac{1}{2^3}(u_1 + 2u_2 + u_3) + \cdots\cdot\cdot \tag{7}$$

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* Read before the American Mathematical Society at its meeting, April 26, 1902.

It turns out, in a large variety of slowly convergent series in which the positive and negative terms are about evenly balanced in number and aggregate value throughout the series, that series (7) converges rapidly and the error made by stopping at any term is less than the last term added. In practical computation this fact will often be sufficiently evident from a simple inspection of the series. The question of finding an upper limit to the error will be considered later. See §5.

2. Illustrative Examples. We will now illustrate the method described in the last section by applying it to some numerical examples, and in doing so we will indicate a number of slight modifications which are often advantageous.

Example I. $S(x) = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots.$

To evaluate this series when $x = 50°$ we may tabulate the work as follows:

n	u_n	$u_n + u_{n+1}$	$u_n + 2u_{n+1} + \cdot\cdot$	$u_n + 3u_{n+1} + \cdot\cdot$	$u_n + 4u_{n+1} + \cdot\cdot$	$u_n + 5u_{n+1} + \cdot\cdot$	$u_n + 6u_{n+1}$
1	**+ .7660**	**+ .2736**	**— .0521**	**— .1256**	**— .0493**	**+ .0308**	**+ .0446**
2	— .4924	— .3257	— .0735	+ .0763	+ .0801	+ .0138	
3	+ .1667	+ .2522	+ .1498	+ .0038	— .0663		
4	+ .0855	— .1024	— .1460	— .0701			
5	— .1878	— .0436	+ .0759				
6	+ .1443	+ .1195					
7	— .0248						

The first terms of the successive columns correspond to the terms of (7).

$S = \frac{1}{2}\times .7660 + \frac{1}{4}\times .2736 - \frac{1}{8}\times .0521 + \cdots + \frac{1}{128}\times .0446 + \cdot\cdot = .4364 +.$

But $S(x) = \frac{x}{2}$, therefore the correct result is .43633. For $x = 10°$ the same series gives a result correct to four places by the use of four terms.

Example II. $S = \frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \cdots\cdots.$

We compute the table as before:

n	u_n	$u_n + u_{n+1}$	$u_n + 2u_{n+1} + \cdot\cdot$	$u_n + 3u_{n+1} + \cdot\cdot$	$u_n + 4u_{n+1} + \cdot\cdot$	$u_n + 5u_{n+1} + \cdot\cdot$
1	**1.4427**	.5325	.3436	.2547	.2026	.1683
2	**— .9102**	— .1889	— .0889	— .0521	— .0343	
3	**.7213**	**.1000**	**.0368**	**.0178**		
4	— .6216	— .0632	— .0190			
5	.5581	.0442				
6	— .5139					

A slight inspection shows that it is best to sum two terms separately, using the terms in heavy type. The terms above those used need not have been computed.

$$S = 1.4427 - .9102 + \left\{ \tfrac{1}{2} \times .7213 + \tfrac{1}{4} \times .1000 + \cdot\cdot \right\} = .9239 + .$$

This result is correct to the third place of decimals, as can be proved by §5.

Example III. The reader may compute

$$S(x) = P_0(\mu) - \tfrac{1}{2} P_2(\mu)x + \tfrac{1 \cdot 3}{2 \cdot 4} P_4(\mu)x^2 - \tfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} P_6(\mu)x^3 + \cdot \cdot \cdot ,$$

for x equal to or near 1, and μ any desired value. (*Cf.* Byerly, *Fourier's Series and Spherical Harmonics*, p. 152.)

Example IV. $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdot \cdot \cdot .$

Since the more slowly the original series converges the more rapidly the resulting series converges, we sum ten terms separately and apply the transformation (7) to the next eleven. We obtain $\frac{\pi}{4} = .785398163$, correct to nine places (*cf.* §5).

Example V. $S = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdot \cdot \cdot, \quad p = 1.001.$

Sum four terms separately, and apply the transformation to the next seven.

$$S = .693309 + R, \qquad R < .000003. \qquad (\S 5.)$$

Example VI. $S = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdot \cdot \cdot, \quad p = 1.001.$

$$S = \frac{2^p}{2^p - 1}\left(1 - \frac{1}{2^p} + \frac{1}{3^p} \quad \cdot \cdot \cdot \cdot\right) = 1000.563 + R, \qquad \text{(Ex. V.)}$$

$$R < .004.$$

Example VII. $S = 1 + \frac{5 \cdot 7}{8 \cdot 10} + \frac{5 \cdot 7 \cdot 9 \cdot 11}{8 \cdot 10 \cdot 12 \cdot 14} + \cdot \cdot \cdot \cdot .$

$$S = 6\left[1 - \frac{5}{6} + \frac{5 \cdot 7}{6 \cdot 8} - \frac{5 \cdot 7 \cdot 9}{6 \cdot 8 \cdot 10} + \cdot \cdot \cdot\right] = 3.341 + R, \qquad R < .006$$

by the use of four terms (*cf.* §5).

Example VIII. $S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} \cdot \cdot \cdot \cdot \cdot .$

Group the terms thus :

$$S = \left(1 + \frac{1}{3}\right) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4} + \cdot \cdot \cdot \cdot .$$

Computing the table as in Ex. I, the fifth column is seen to consist of positive terms. It will not be useful to obtain another column. By (5) :

$$S = \tfrac{1}{2} \times 1.333 + \tfrac{1}{4} \times .833 + \tfrac{1}{8} \times .676 + \tfrac{1}{16} \times .612 + \tfrac{1}{16}[.593 + .013 + .030$$

$$+ .003 + \cdot \ \cdot] = 1.037 + \cdot$$

Here the degree of accuracy is not evident. There is strong probability that the result is much closer to the true value than could have been reached by simply adding terms. This example illustrates the limitations of the method; also how it will usually become evident if the method cannot be profitably employed in its entirety, and how it may still be more or less useful in a modified form.

3. Proof of Convergence. We will now complete the proof of (7) by showing that the remainder R_k of (6) approaches zero as k becomes infinite.

$$R_k = R_k' + R_k'' + R_k''',$$

where

$$R'_k = \frac{1}{2^k}\Big[u_1 + (1+k)u_2 + \Big(1 + k + \frac{k(k-1)}{1\cdot 2}\Big)u_3 + \cdot\cdot\cdot$$

$$+ \Big(1 + k + \frac{k(k-1)}{1\cdot 2} + \cdot\cdot\cdot + \frac{k(k-1)\cdot\cdot(k-r+2)}{1\cdot 2\ .\ .\ .\ (r-1)}\Big)u_r\Big]\cdot$$

$$R''_k = \frac{1}{2^k}\Big[\Big(1 + k + \frac{k(k-1)}{1\cdot 2} + \cdot\cdot\cdot + \frac{k(k-1)\cdot\cdot\cdot(k-r+1)}{1\cdot 2\cdot\cdot\cdot r}\Big)u_{r+1} + \cdots$$

$$+ \Big(1 + k + \frac{k(k-1)}{1\cdot 2} + \cdot\cdot\cdot + \frac{k(k-1)}{1\cdot 2} + k\Big)u_k\Big],$$

$$R'''_k = u_{k+1} + u_{k+2} + \cdot\cdot\cdot\cdot.$$

Our theorem will be proved if we can show that a positive ϵ having been chosen at pleasure, a positive m exists such that when $k > m$:

$$|R'_k| < \frac{\epsilon}{3}, \qquad |R''_k| < \frac{\epsilon}{3}, \qquad |R'''_k| < \frac{\epsilon}{3}\cdot$$

Since the u-series converges, m''' can be chosen so that $|R'''_k| < \epsilon/3$ when $k > m'''$.

Similarly choose m'' so that when $k > r > m''$:

$$|u_{r+1} + u_{r+2} + \cdot\cdot\cdot\cdot\cdot\cdot\ u_k| < \epsilon/3.$$

R''_k is of the form:

$$\eta_{r+1}u_{r+1} + \eta_{r+2}u_{r+2} + \cdot\cdot\cdot\cdot\cdot\cdot + \eta_k u_k,$$

where

$$0 < \eta_{r+1} < \eta_{r+2} < \cdot\cdot\cdot\cdot\cdot\cdot < \eta_k < 1.$$

Hence, by a well-known lemma of Abel's,* when $k > r > m''$,

$$|R''_k| < \epsilon/3.$$

Let M be a positive constant greater than the absolute value of any term of the u-series. Then

$$|R'_k| < \frac{Mr}{2^k}\Big[1 + k + \frac{k(k-1)}{1\cdot 2} + \cdot\cdot\cdot + \frac{k(k-1)\cdot\cdot\cdot(k-r+2)}{1\cdot 2\cdot\cdot\cdot(r-1)}\Big]$$

$$< \frac{Mr^2}{2^k}\Big[\frac{k(k-1)\cdot\cdot\cdot\cdot(k-r+2)}{1\cdot 2\cdot\cdot\cdot\cdot(r-1)}\Big], \qquad \text{if } k > 2r,$$

$$< \frac{Mr^2}{2^k}\,k^{r-1}.$$

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* *Cf.* for instance Tannéry: *Théorie des fonctions d'une variable*, p. 95.

Put $k = r^2$, then $\qquad |R'_k| < \dfrac{Mr^{2r}}{2^{r^2}} = M\left(\dfrac{r^2}{2^r}\right)^r.$

$$\text{But} \quad \lim_{r=\infty} \frac{r^2}{2^r} = 0. \qquad \therefore \lim_{r=\infty} \left(\frac{r^2}{2^r}\right)^r = 0. \qquad \therefore \lim_{k=\infty} R'_k = 0.$$

Therefore m' can be chosen so that $|R'_k| < \epsilon/3$ when $k > m'$.

By taking m greater than the largest of the three quantities m', m'', m''', the truth of the theorem follows at once.

4. A Shorter Method of Computation. By subtracting R_k, in the form given in §3, from S we obtain :

$$S = \frac{1}{2^k}\left[\lambda_1 u_1 + \lambda_2 u_2 + \cdot \cdot \cdot \; \lambda_k u_k\right] + R_k, \tag{8}$$

where

$$\lambda_r = 1 + k + \frac{k(k-1)}{1 \cdot 2} + \cdot \cdot \cdot \frac{k(k-1) \cdot \cdot \cdot (r+1)}{1 \cdot 2 \cdot \cdot \cdot (k-r)}. \tag{9}$$

The λ's are the same for all series and may be computed once for all. A few sets are tabulated :

k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}
4	15	11	5	1							
7	128	120	99	64	29	8	1				
11	2047	2036	1981	1816	1486	1024	562	232	67	12	1

By the use of this table Ex. II, §2 would be computed as follows :

$$S = 1.4427 - .9102 + \frac{1}{2^4}(15 \times .7213 - 11 \times .6216 + 5 \times .5581 - .5139) + R$$
$$= .9237 + R.$$

Unless we know from other considerations (§5) an upper limit to R the method used in §2 is usually more convenient, since in that method we can observe the rate of convergence.

5. Determination of an Upper Limit for the Error. Let us confine ourselves henceforth to the alternating series :

$$S = v_1 - v_2 + v_3 - v_4 + \cdots, \qquad v_n > 0. \tag{10}$$

Form the successive orders of differences of the absolute value series $v_1 + v_2 + v_3 + \cdots$ If we denote the general term by $\Delta^r v_n$, the series (7) becomes:

$$S = \frac{1}{2} v_1 - \frac{1}{2^2} \Delta^1 v_1 + \frac{1}{2^3} \Delta^2 v_1 - \cdots\cdot + (-1)^{r-1} \frac{1}{2^r} \Delta^{r-1} v_1 + \cdots\cdot \tag{11}$$

In some cases the differences can be computed in general terms. Thus let:

$$S = 1 - \frac{a}{b} + \frac{a(a+1)}{b(b+1)} - \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \cdots\cdot, \qquad a < b. \tag{12}$$

Computing the differences and substituting in (11):

$$S = \frac{1}{2} + \frac{1}{2^2} \frac{b-a}{b} + \frac{1}{2^3} \frac{(b-a)(b-a+1)}{b(b+1)} + \cdots . \tag{13}$$

The ratio of any term to the preceding term is less than $\frac{1}{2}$. Hence the error made by stopping at any term is less than the last term added. The series in §2, Exs. IV and VII are of this form.

Theorem. *If there is a function $f(x)$ which has continuous derivatives of the first $k + 1$ orders when $x \geqq 1$; and if the derivatives of even order are positive, those of odd order negative; and $f(n) = v_n = (-1)^{n+1} u_n > 0$; then*

$$|R_k| < \frac{1}{2^k} |u_1|.$$

Proof. Form the sequence

$$f(x_0), \quad f(x_0 + \Delta x), \quad f(x_0 + 2\Delta x), \cdots\cdot$$

It can be proved* that the rth difference quotient $\Delta^r f(x)/\Delta x$ converges uniformly to the value $d^r f(x)/dx^r$ as Δx approaches zero. Hence a positive number δ may be chosen so that for all values of $x \geqq 1$, when $\Delta x < \delta$, $\Delta^r f(x)/\Delta x^r$ and $d^r f(x)/dx^r$ have the same sign. Now choose $\Delta x < \delta$ so that Δx is an aliquot part of unity, *i. e.*, $m \Delta x = 1$. Then the v-series may be formed by taking every mth term of the above sequence, if x_0 is properly chosen. It is easily shown that any difference of the v-series is the sum of positive multiples of differences of the same order of this series. Hence $\Delta^r v_n$ is positive when r is even and negative when r is odd. It follows that

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* *Cf.* for instance, Harnack, *Die Elemente der Differential- und Integralrechnung*, §32, where the proof can easily be made to meet the question as to the uniformity of the convergence.

$$|\Delta^k v_n| > |\Delta^k v_{n+1}|.$$

But, from (6),

$$R_k = \frac{1}{2^k}\left(\Delta^k v_1 - \Delta^k v_2 + \Delta^k v_3 \cdot \cdot \cdot \cdot\right).$$

Hence

$$|R_k| < \frac{1}{2^k}\,|\Delta^k v_1| < \frac{1}{2^k}\,|u_1|\,.$$

By the use of this theorem we can prove that the results of Exs. II, IV, V, VI are correct.

6. Application to Divergent Series. If our method be applied to certain divergent series it renders them convergent. This fact becomes of peculiar interest when we apply the method to power series, as we thus get an analytic continuation of the function originally represented. But similar results may be reached by the method of conformal transformation.*

As this treatment of this phase of the subject is simpler, we will not enter upon the discussion here.

HARVARD UNIVERSITY,
CAMBRIDGE, MASSACHUSETTS,
APRIL, 1902.

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* Painlevé has given a general statement of the method; Paris Thesis, 1887; *Annales de la Faculté des Sciences de Toulouse*, vol. 2 (1888), p. B.1. A number of special developments are worked out in detail by E. Lindelöf, Acta societatis scientiarum Fennicae, vol. 24 (1898), No. 7.

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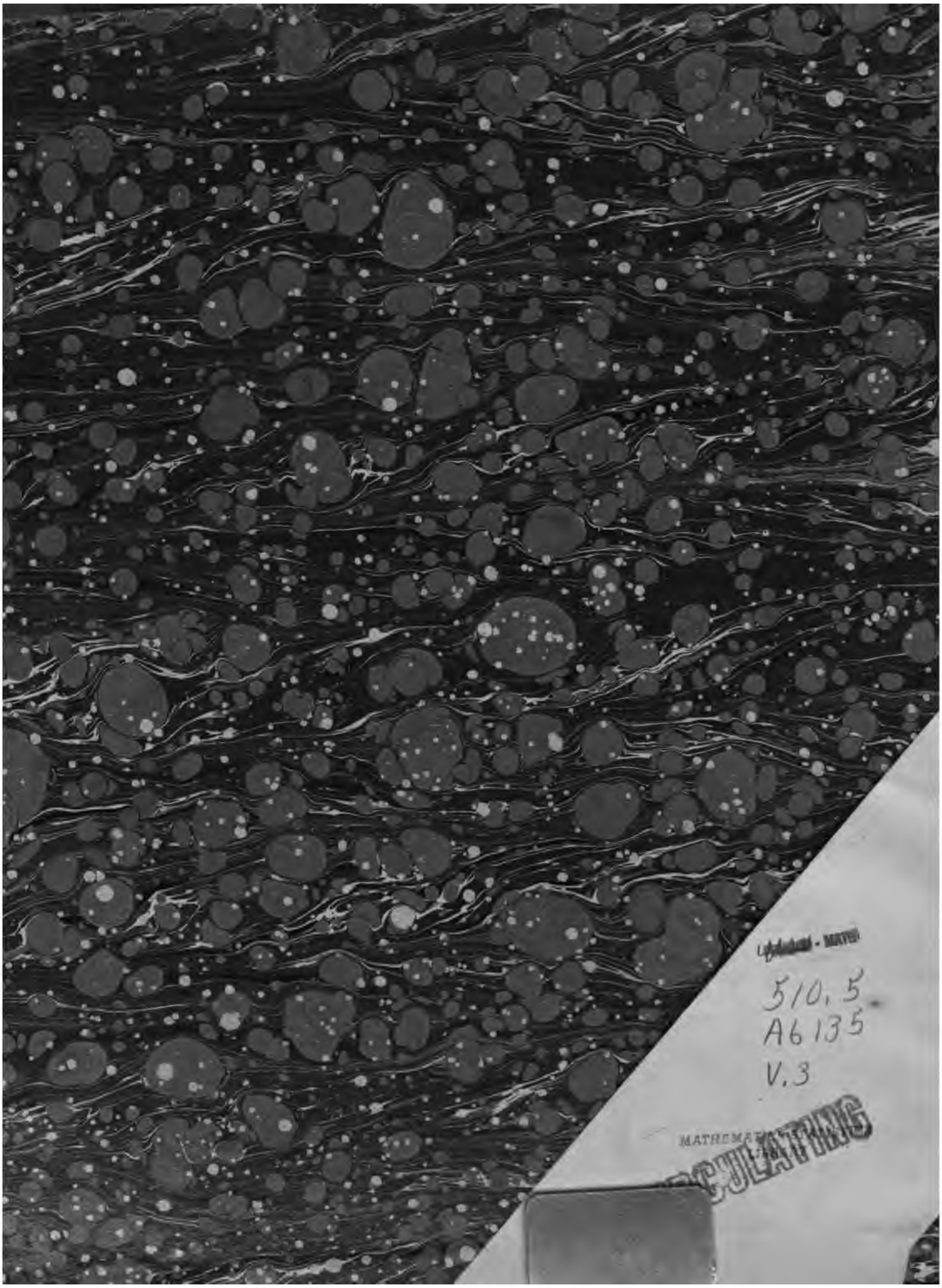

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